# History of Modern Mathematics

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## Article 3: Irrational and Transcendent Numbers

The sixteenth century saw the final acceptance of negative numbers, integral and fractional. The seventeenth century saw decimal fractions with the modern notation quite generally used by mathematicians. The next hundred years saw the imaginary become a powerful tool in the hands of De Moivre, and especially of Euler. For the nineteenth century it remained to complete the theory of complex numbers, to separate irrationals into algebraic and transcendent, to prove the existence of transcendent numbers, and to make a scientific study of a subject which had remained almost dormant since Euclid, the theory of irrationals. The year 1872 saw the publication of the theories of Weierstrass (by his pupil Kossak), Heine (Crelle, 74), G. Cantor (Annalen, 5), and Dedekind. Méray had taken in 1869 the same point of departure as Heine, but the theory is generally referred to the year 1872. Weierstrass's method has been completely set forth by Pincherle (1880), and Dedekind's has received additional prominence through the author's later work (1888) and the recent indorsement by Tannery (1894). Weierstrass, Cantor, and Heine base their theories on infinite series, while Dedekind founds his on the idea of a cut (Schnitt) in the system of real numbers, separating all rational numbers into two groups having certain characteristic properties. The subject has received later contributions at the hands of Weierstrass, Kronecker (Crelle, 101), and Méray.

Continued Fractions, closely related to irrational numbers and due to Cataldi, 1613),1 received attention at the hands of Euler, and at the opening of the nineteenth century were brought into prominence through the writings of Lagrange. Other noteworthy contributions have been made by Druckenmüller (1837), Kunze (1857), Lemke (1870), and Günther (1872). Ramus (1855) first connected the subject with determinants, resulting, with the subsequent contributions of Heine, Möbius, and Günther, in the theory of Kettenbruchdeterminanten. Dirichlet also added to the general theory, as have numerous contributors to the applications of the subject.

Transcendent Numbers2 were first distinguished from algebraic irrationals by Kronecker. Lambert proved (1761) that π cannot be rational, and that en (n being a rational number) is irrational, a proof, however, which left much to be desired.

Legendre (1794) completed Lambert's proof, and showed that π is not the square root of a rational number. Liouville (1840) showed that neither e nor e2 can be a root of an integral quadratic equation. But the existence of transcendent numbers was first established by Liouville (1844, 1851), the proof being subsequently displaced by G. Cantor's (1873). Hermite (1873) first proved e transcendent, and Lindemann (1882), starting from Hermite's conclusions, showed the same for π. Lindemann's proof was much simplified by Weierstrass (1885), still further by Hilbert (1893), and has finally been made elementary by Hurwitz and Gordan.

1 But see Favaro, A., Notizie storiche sulle frazioni continue dal secolo decimoterzo al deci mosettimo, Boncompagni's Bulletino, Vol. VII, 1874, pp. 451, 533.

2 Klein, F., Vorträge über ausgewählte Fragen der Elementargeometrie, 1895, p. 38; Bachmann, P., Vorlesungen über die Natur der Irrationalzahlen, 1892.