History of Modern Mathematics

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ARTICLE 4: COMPLEX NUMBERS

The Theory of Complex Numbers1 may be said to have attracted attention as early as the sixteenth century in the recognition, by the Italian algebraists, of imaginary or impossible roots. In the seventeenth century Descartes distin- guished between real and imaginary roots, and the eighteenth saw the labors of De Moivre and Euler. To De Moivre is due (1730) the well-known formula which bears his name, $(\cos \square + i \sin \square)n = \cos n\square + i \sin n\square$, and to Euler (1748) the formula $\cos \square + i \sin \square = e\square\, i$.

The geometric notion of complex quantity now arose, and as a result the theory of complex numbers received a notable expansion. The idea of the graphic representation of complex numbers had appeared, however, as early as 1685, in Wallis's De Algebra tractatus. In the eighteenth century Kühn (1750) and Wessel (about 1795) made decided advances towards the present theory. Wessel's memoir appeared in the Proceedings of the Copenhagen Academy for 1799, and is exceedingly clear and complete, even in comparison with modern works. He also considers the sphere, and gives a quaternion theory from which he develops a complete spherical trigonometry. In 1804 the Abbé Buée independently came upon the same idea which Wallis had suggested, that $\pm\sqrt{-1}$ should represent a unit line, and its negative, perpendicular to the real axis. Buée's paper was not published until 1806, in which year Argand also issued a pamphlet on the same subject. It is to Argand's essay that the scientific foundation for the graphic representation of complex numbers is now generally referred. Nevertheless, in 1831 Gauss found the theory quite unknown, and in 1832 published his chief memoir on the subject, thus bringing it prominently before the mathematical world. Mention should also be made of an excellent little treatise by Mourey (1828), in which the foundations for the theory of directional numbers are scientifically laid. The general acceptance of the theory is not a little due to the labors of Cauchy and Abel, and especially the latter, who was the first to boldly use complex numbers with a success that is well known.

The common terms used in the theory are chiefly due to the founders. Argand called $\cos \square + i \sin \square$ the "direction factor", and $r = \sqrt{a2 + b2}$ the "modulus"; Cauchy (1828) called $\cos \square + i \sin \square$ the "reduced form" (l'expression réduite); Gauss used i for $\sqrt{-1}$, introduced the term "complex number" for a + bi, and called a2 + b2 the

"norm." The expression "direction coefficient", often used for cos $\square$ + i sin $\square$, is due to Hankel (1867), and "absolute value," for "modulus," is due to Weierstrass.

Following Cauchy and Gauss have come a number of contributors of high rank, of whom the following may be especially mentioned: Kummer (1844), Kronecker (1845), Scheffler (1845, 1851, 1880), Bellavitis (1835, 1852), Peacock (1845), and De Morgan (1849). Möbius must also be mentioned for his numerous memoirs on the geometric applications of complex numbers, and Dirichlet for the expansion of the theory to include primes, congruences, reciprocity, etc., as in the case of real numbers.

Other types[2] have been studied, besides the familiar a + bi, in which i is the root of $x^2 + 1 = 0$. Thus Eisenstein has studied the type a + bj , j being a complex root of $x^3 - 1 = 0$. Similarly, complex types have been derived from $x^k - 1 = 0$ (k prime). This generalization is largely due to Kummer, to whom is also due the theory of Ideal Numbers,[3] which has recently been simplified by Klein (1893) from the point of view of geometry. A further complex theory is due to Galois, the basis being the imaginary roots of an irreducible congruence, F (x) = 0 (mod p, a prime). The late writers (from 1884) on the general theory include Weierstrass, Schwarz, Dedekind, Hölder, Berloty, Poincaré, Study, and Macfarlane.

[1] Riecke, F., Die Rechnung mit Richtungszahlen, 1856, p. 161; Hankel, H., Theorie der komplexen Zahlensysteme, Leipzig, 1867; Holzmüller, G., Theorie der isogonalen Verwandtschaften, 1882, p. 21; Macfarlane, A., The Imaginary of Algebra, Proceedings of American Association 1892, p. 33; Baltzer, R., Einführung der komplexen Zahlen, Crelle, 1882; Stolz, O., Vorlesungen über allgemeine Arithmetik, 2. Theil, Leipzig, 1886.

[2] Chapman, C. H., Weierstrass and Dedekind on General Complex Numbers, in Bulletin New York Mathematical Society, Vol. I, p. 150; Study, E., Aeltere und neuere Untersuchungen über Systeme complexer Zahlen, Mathematical Papers Chicago Congress, p. 367; bibliography, p. 381.

[3] Klein, F., Evanston Lectures, Lect. VIII.