ARTICLE 12: INFINITE SERIES

The Theory of Infinite Series1 in its historical development has been divided by Reiff into three periods: (1) the period of Newton and Leibniz, that of its introduction; (2) that of Euler, the formal period; (3) the modern, that of the scientific investigation of the validity of infinite series, a period beginning with Gauss. This critical period begins with the publication of Gauss's celebrated memoir on the series in 1812. Euler had already considered this series, but Gauss was the first to master it, and under the name "hypergeometric series" (due to Pfaff) it has since occupied the attention of Jacobi, Kummer, Schwarz, Cayley, Goursat, and numerous others. The particular series is not so important as is the standard of criticism which Gauss set up, embodying the simpler criteria of convergence and the questions of remainders and the range of convergence.

Gauss's contributions were not at once appreciated, and the next to call attention to the subject was Cauchy (1821), who may be considered the founder of the theory of convergence and divergence of series. He was one of the first to insist on strict tests of convergence; he showed that if two series are convergent their product is not necessarily so; and with him begins the discovery of effective criteria of convergence and divergence. It should be mentioned, however, that these terms had been introduced long before by Gregory (1668), that Euler and Gauss had given various criteria, and that Maclaurin had anticipated a few of Cauchy's discoveries. Cauchy advanced the theory of power series by his expansion of a complex function in such a form. His test for convergence is still one of the most satisfactory when the integration involved is possible.

Abel was the next important contributor. In his memoir (1826) on the series he corrected certain of Cauchy's conclusions, and gave a completely scientific summation of the series for complex values of m and x. He was emphatic against the reckless use of series, and showed the necessity of considering the subject of continuity in questions of convergence.

Cauchy's methods led to special rather than general criteria, and the same may be said of Raabe (1832), who made the first elaborate investigation of the subject, of De

Morgan (from 1842), whose logarithmic test DuBois-Reymond (1873) and Pringsheim (1889) have shown to fail within a certain region; of Bertrand (1842), Bonnet (1843), Malmsten (1846, 1847, the latter without integration); Stokes (1847), Paucker (1852), Tchébichef (1852), and Arndt (1853). General criteria began with Kummer (1835), and have been studied by Eisenstein (1847), Weierstrass in his various contributions to the theory of functions, Dini (1867), DuBois-Reymond (1873), and many others. Pringsheim's (from 1889) memoirs present the most complete general theory.

The Theory of Uniform Convergence was treated by Cauchy (1821), his limitations being pointed out by Abel, but the first to attack it successfully were Stokes and Seidel (1847-48). Cauchy took up the problem again (1853), acknowledging Abel's criticism, and reaching the same conclusions which Stokes had already found. Thomé used the doctrine (1866), but there was great delay in recognizing the importance of distinguishing between uniform and non-uniform convergence, in spite of the demands of the theory of functions.

Semi-Convergent Series were studied by Poisson (1823), who also gave a general form for the remainder of the Maclaurin formula. The most important solution of the problem is due, however, to Jacobi (1834), who attacked the question of the remainder from a different standpoint and reached a different formula. This expression was also worked out, and another one given, by Malmsten (1847). Schlömilch (Zeitschrift, Vol. I, p. 192, 1856) also improved Jacobi's remainder, and showed the relation between the remainder and Bernoulli's function F(x) = 1n + 2n + · · · + (x – 1)n. Genocchi (1852) has further contributed to the theory.

Among the early writers was Wronski, whose "loi supr ˆeme" (1815) was hardly recognized until Cayley (1873) brought it into prominence. Transon (1874), Ch. Lagrange (1884), Echols, and Dickstein[2] have published of late various memoirs on the subject.

Interpolation Formulas have been given by various writers from Newton to the present time. Lagrange's theorem is well known, although Euler had already given an analogous form, as are also Olivier's formula (1827), and those of Minding (1830), Cauchy (1837), Jacobi (1845), Grunert (1850, 1853), Christoffel (1858), and Mehler (1864).

Fourier's Series[3] were being investigated as the result of physical considerations at the same time that Gauss, Abel, and Cauchy were working out the theory of infinite series. Series for the expansion of sines and cosines, of multiple arcs in powers of the sine and cosine of the arc had been treated by Jakob Bernoulli (1702) and his brother Johann (1701) and still earlier by Viète. Euler and Lagrange had simplified the subject, as have, more recently, Poinsot, Schröter, Glaisher, and Kummer. Fourier (1807) set for himself a different problem, to expand a given function of x in terms of the sines or cosines of multiples of x, a problem which he embodied in his Théorie analytique de la Chaleur (1822). Euler had already given the formulas for determining the coefficients in the series; and Lagrange had passed over them without recognizing their value, but Fourier was the first to assert and attempt to prove the general theorem. Poisson (1820-23) also attacked the problem from a different standpoint. Fourier did not, however, settle the question of convergence of his series, a matter left for Cauchy (1826) to attempt and for Dirichlet (1829) to handle in a thoroughly scientific manner. Dirichlet's treatment (Crelle, 1829), while bringing the theory of trigonometric series to a temporary conclusion, has been the subject of criticism and improvement by Riemann (1854), Heine, Lipschitz, Schläfli, and DuBois-Reymond. Among other prominent contributors to the theory of trigonometric and Fourier series have been Dini, Hermite, Halphen, Krause, Byerly and Appell.

1 Cantor, M., Geschichte der Mathematik, Vol. III, pp. 53, 71; Reiff, R., Geschichte der unendlichen Reihen, T?ubingen, 1889; Ca jori, F., Bulletin New York Mathematical Society, Vol. I, p. 184; History of Teaching of Mathematics in United States, p. 361.

2 Bibliotheca Mathematica, 1892-94; historical.

3 Historical Summary by Bˆocher, Chap. IX of Byerly's Fourier's Series and Spherical Harmonics, Boston, 1893; Sachse, A., Essai historique sur la représentation d'une fonction . . . par une série trigonométrique. Bulletin des Sciences mathématiques, Part I, 1880, pp. 43, 83.