Article 15: Analytic Geometry

The History of Geometry1 may be roughly divided into the four periods: (1) The synthetic geometry of the Greeks, practically closing with Archimedes; (2) The birth of analytic geometry, in which the synthetic geometry of Guldin, Desargues, Kepler, and Roberval merged into the coordinate geometry of Descartes and Fermat; (3) 1650 to 1800, characterized by the application of the calculus to geometry, and including the names of Newton, Leibnitz, the Bernoullis, Clairaut, Maclaurin, Euler, and Lagrange, each an analyst rather than a geometer; (4) The nineteenth century, the renaissance of pure geometry, characterized by the descriptive geometry of Monge, the modern synthetic of Poncelet, Steiner, von Staudt, and Cremona, the modern analytic founded by Plücker, the non-Euclidean hypothesis of Lobachevsky and Bolyai, and the more elementary geometry of the triangle founded by Lemoine. It is quite impossible to draw the line between the analytic and the synthetic geometry of the nineteenth century, in their historical development, and Arts. 15 and 16 should be read together.

The Analytic Geometry which Descartes gave to the world in 1637 was confined to plane curves, and the various important properties common to all algebraic curves were soon discovered. To the theory Newton contributed three celebrated theorems on the Enumeratio linearum tertii ordinis2 (1706), while others are due to Cotes (1722), Maclaurin, and Waring (1762, 1772, etc.). The scientific foundations of the theory of plane curves may be ascribed, however, to Euler (1748) and Cramer (1750). Euler distinguished between algebraic and transcendent curves, and attempted a classification of the former. Cramer is well known for the "paradox" which bears his name, an obstacle which Lamé (1818) finally removed from the theory. To Cramer is also due an attempt to put the theory of singularities of algebraic curves on a scientific foundation, although in a modern geometric sense the theory was first treated by Poncelet.

Meanwhile the study of surfaces was becoming prominent. Descartes had suggested that his geometry could be extended to three-dimensional space, Wren (1669) had discovered the two systems of generating lines on the hyperboloid of one sheet, and Parent (1700) had referred a surface to three coordinate planes. The geometry of three dimensions began to assume definite shape, however, in a memoir of

Clairaut's (1731), in which, at the age of sixteen, he solved with rare elegance many of the problems relating to curves of double curvature. Euler (1760) laid the foundations for the analytic theory of curvature of surfaces, attempting the classification of those of the second degree as the ancients had classified curves of the second order. Monge, Hachette, and other members of that school entered into the study of surfaces with great zeal. Monge introduced the notion of families of surfaces, and discovered the relation between the theory of surfaces and the integration of partial differential equations, enabling each to be advantageously viewed from the standpoint of the other. The theory of surfaces has attracted a long list of contributors in the nineteenth century, including most of the geometers whose names are mentioned in the present article.3

Möbius began his contributions to geometry in 1823, and four years later published his Barycentrische Calcül. In this great work he introduced homogeneous coordinates with the attendant symmetry of geometric formulas, the scientific exposition of the principle of signs in geometry, and the establishment of the principle of geometric correspondence simple and multiple. He also (1852) summed up the classification of cubic curves, a service rendered by Zeuthen (1874) for quartics. To the period of Möbius also belong Bobillier (1827), who first used trilinear coordinates, and Bellavitis, whose contributions to analytic geometry were extensive. Gergonne's labors are mentioned in the next article.

Of all modern contributors to analytic geometry, Plücker stands foremost. In 1828 he published the first volume of his Analytisch-geometrische Entwickelungen, in which appeared the modern abridged notation, and which marks the beginning of a new era for analytic geometry. In the second volume (1831) he sets forth the present analytic form of the principle of duality. To him is due (1833) the general treatment of foci for curves of higher degree, and the complete classification of plane cubic curves (1835) which had been so frequently tried before him. He also gave (1839) an enumeration of plane curves of the fourth order, which Bragelogne and Euler had attempted. In 1842 he gave his celebrated "six equations" by which he showed that the characteristics of a curve (order, class, number of double points, number of cusps, number of double tangents, and number of inflections) are known when any three are given. To him is also due the first scientific dual definition of a curve, a system of tangential coordinates, and an investigation of the question of double tangents, a question further elaborated by Cayley (1847, 1858), Hesse (1847), Salmon (1858), and Dersch (1874). The theory of ruled surfaces, opened by Monge, was also extended by him. Possibly the greatest service rendered by Pl ücker was the introduction of the straight line as a space element, his first contribution (1865) being followed by his well-known treatise on the subject (1868-69). In this work he treats certain general properties of complexes, congruences, and ruled surfaces, as well as special properties of linear complexes and congruences, subjects also considered by Kummer and by Klein and others of the modern school. It is not a little due to Plücker that the concept of 4- and hence n-dimensional space, already suggested by Lagrange and Gauss, became the subject of later research. Riemann, Helmholtz, Lipschitz, Kronecker, Klein, Lie, Veronese, Cayley, d'Ovidio, and many others have elaborated the theory. The regular hypersolids in 4-dimensional space have been the subject of special study by Scheffler, Rudel, Hoppe, Schlegel, and Stringham.

Among Jacobi's contributions is the consideration (1836) of curves and groups of points resulting from the intersection of algebraic surfaces, a subject carried forward by Reye (1869). To Jacobi is also due the conformal representation of the ellipsoid on a plane, a treatment completed by Schering (1858). The number of examples of conformal representation of surfaces on planes or on spheres has been increased by Schwarz (1869) and Amstein (1872).

In 1844 Hesse, whose contributions to geometry in general are both numerous and valuable, gave the complete theory of inflections of a curve, and introduced the so-called Hessian curve as the first instance of a covariant of a ternary form. He also contributed to the theory of curves of the third order, and generalized the Pascal and Brianchon theorems on a spherical surface. Hesse's methods have recently been elaborated by Gundelfinger (1894).

Besides contributing extensively to synthetic geometry, Chasles added to the theory of curves of the third and fourth degrees. In the method of characteristics which he worked out may be found the first trace of the Abzählende Geometrie4 which has been developed by Jonquières, Halphen (1875), and Schubert (1876, 1879), and to which Clebsch, Lindemann, and Hurwitz have also contributed. The general theory of correspondence starts with Geometry, and Chasles (1864) undertook the first special researches on the correspondence of algebraic curves, limiting his investigations, however, to curves of deficiency zero. Cayley (1866) carried this theory to curves of higher deficiency, and Brill (from 1873) completed the theory.

Cayley's5 influence on geometry was very great. He early carried on Plücker's consideration of singularities of a curve, and showed (1864, 1866) that every singularity may be considered as compounded of ordinary singularities so that the "six equations" apply to a curve with any singularities whatsoever. He thus opened a field for the later investigations of Noether, Zeuthen, Halphen, and H. J. S. Smith.

Cayley's theorems on the intersection of curves (1843) and the determination of self-corresponding points for algebraic correspondences of a simple kind are fundamental in the present theory, subjects to which Bacharach, Brill, and Noether have also contributed extensively. Cayley added much to the theories of rational transformation and correspondence, showing the distinction between the theory of transformation of spaces and that of correspondence of loci. His investigations on the bitangents of plane curves, and in particular on the twenty-eight bitangents of a non-singular quartic, his developments of Plücker's conception of foci, his discussion of the osculating conics of curves and of the sextactic points on a plane curve, the geometric theory of the invariants and covariants of plane curves, are all noteworthy. He was the first to announce (1849) the twenty-seven lines which lie on a cubic surface, he extended Salmon's theory of reciprocal surfaces, and treated (1869) the classification of cubic surfaces, a subject already discussed by Schläfli. He also contributed to the theory of scrolls (skew-ruled surfaces), orthogonal systems of surfaces, the wave surface, etc., and was the first to reach (1845) any very general results in the theory of curves of double curvature, a theory in which the next great advance was made (1882) by Halphen and Noether. Among Cayley's other contributions to geometry is his theory of the Absolute, a figure in connection with which all metrical properties of a figure are considered.

Clebsch6 was also prominent in the study of curves and surfaces. He first applied the algebra of linear transformation to geometry. He emphasized the idea of deficiency (Geschlecht) of a curve, a notion which dates back to Abel, and applied the theory of elliptic and Abelian functions to geometry, using it for the study of curves. Clebsch (1872) investigated the shapes of surfaces of the third order. Following him, Klein attacked the problem of determining all possible forms of such surfaces, and established the fact that by the principle of continuity all forms of real surfaces of the third order can be derived from the particular surface having four real conical points. Zeuthen (1874) has discussed the various forms of plane curves of the fourth order, showing the relation between his results and those of Klein on cubic surfaces. Attempts have been made to extend the subject to curves of the nth order, but no general classification has been made. Quartic surfaces have been studied by Rohn (1887) but without a complete enumeration, and the same writer has contributed (1881) to the theory of Kummer surfaces.

Lie has adopted Plücker's generalized space element and extended the theory. His sphere geometry treats the subject from the higher standpoint of six homogeneous coordinates, as distinguished from the elementary sphere geometry with but five and characterized by the conformal group, a geometry studied by Darboux. Lie's theory of contact transformations, with its application to differential equations, his line and sphere complexes, and his work on minimum surfaces are all prominent.

Of great help in the study of curves and surfaces and of the theory of functions are the models prepared by Dyck, Brill, O. Henrici, Schwarz, Klein, Schönflies, Kummer, and others.7

The Theory of Minimum Surfaces has been developed along with the analytic geometry in general. Lagrange (1760-61) gave the equation of the minimum surface through a given contour, and Meusnier (1776, published in 1785) also studied the question. But from this time on for half a century little that is noteworthy was done, save by Poisson (1813) as to certain imaginary surfaces. Monge (1784) and Legendre (1787) connected the study of surfaces with that of differential equations, but this did not immediately affect this question. Scherk (1835) added a number of important results, and first applied the labors of Monge and Legendre to the theory. Catalan (1842), Björling (1844), and Dini (1865) have added to the subject. But the most prominent contributors have been Bonnet, Schwarz, Darboux, and Weierstrass. Bonnet (from 1853) has set forth a new system of formulas relative to the general theory of surfaces, and completely solved the problem of determining the minimum surface through any curve and admitting in each point of this curve a given tangent plane, Weierstrass (1866) has contributed several fundamental theorems, has shown how to find all of the real algebraic minimum surfaces, and has shown the connection between the theory of functions of an imaginary variable and the theory of minimum surfaces.

1 Loria, G., Il passato e il presente delle principali teorie geometriche. Memorie Accademia Torino, 1887; translated into German by F. Schutte under the title Die hauptsächlichsten Theorien der Geometrie in ihrer früheren und heutigen Entwickelung, Leipzig, 1888; Chasles, M., Aperçu historique sur l'origine et le développement des méthodes en Géométrie, 1889; Chasles, M., Rapport sur les Progrès de la Géométrie, Paris, 1870; Cayley, A., Curves, Encyclopædia Britannica; Klein, F., Evanston Lectures on Mathematics, New York, 1894; A. V. Braunmühl, Historische Studie über die organische Erzeugung ebener Curven, Dyck's Katalog mathematischer Modelle, 1892.

2 Ball, W. W. R., On Newton's classification of cubic curves. Transactions of London Mathematical Society, 1891, p. 104.

3 For details see Loria, Il passato e il presente, etc.

4 Loria, G., Notizie storiche sulla Geometria numerativa. Bibliotheca Mathematica, 1888, pp. 39, 67; 1889, p. 23.

5 Biographical Notice in Cayley's Collected papers, Vol. VIII.

6 Klein, Evanston Lectures, Lect. I.

7 Dyck, W., Katalog mathematischer und mathematisch-physikalischer Modelle, München, 1892; Deutsche Universitätsausstellung, Mathematical Papers of Chicago Congress, p. 49.