By David Eugene Smith

Article 16: Modern Geometry

Descriptive1, Projective, and Modern Synthetic Geometry are so interwoven in their historic development that it is even more difficult to separate them from one another than from the analytic geometry just mentioned. Monge had been in possession of his theory for over thirty years before the publication of his Géométrie Descriptive (1800), a delay due to the jealous desire of the military authorities to keep the valuable secret. It is true that certain of its features can be traced back to Desargues, Taylor, Lambert, and Frézier, but it was Monge who worked it out in detail as a science, although Lacroix (1795), inspired by Monge's lectures in the École Polytechnique, published the first work on the subject. After Monge's work appeared, Hachette (1812, 1818, 1821) added materially to its symmetry, subsequent French contributors being Leroy (1842), Olivier (from 1845), de la Gournerie (from 1860), Vallée, de Fourcy, Adhémar, and others. In Germany leading contributors have been Ziegler (1843), Anger (1858), and especially Fiedler (3d edn. 1883-88) and Wiener (1884-87). At this period Monge by no means confined himself to the descriptive geometry. So marked were his labors in the analytic geometry that he has been called the father of the modern theory. He also set forth the fundamental theorem of reciprocal polars, though not in modern language, gave some treatment of ruled surfaces, and extended the theory of polars to quadrics.2

Monge and his school concerned themselves especially with the relations of form, and particularly with those of surfaces and curves in a space of three dimensions. Inspired by the general activity of the period, but following rather the steps of Desargues and Pascal, Carnot treated chiefly the metrical relations of figures. In particular he investigated these relations as connected with the theory of transversals, a theory whose fundamental property of a four-rayed pencil goes back to Pappos, and which, though revived by Desargues, was set forth for the first time in its general form in Carnot's Géométrie de Position (1803), and supplemented in his Théorie des Transversales (1806). In these works he introduced negative magnitudes, the general quadrilateral and quadrangle, and numerous other generalizations of value to the elementary geometry of to-day. But although Carnot's work was important and many details are now commonplace, neither the name of the theory nor the method employed have endured. The present Geometry of Position (Geometrie der Lage) has little in common with Carnot's Géométrie de Position.

Projective Geometry had its origin somewhat later than the period of Monge and Carnot. Newton had discovered that all curves of the third order can be derived by central pro jection from five fundamental types. But in spite of this fact the theory attracted so little attention for over a century that its origin is generally ascribed to Poncelet. A prisoner in the Russian campaign, confined at Saratoff on the Volga (1812-14), "privé," as he says, "de toute espèce de livres et de secours, surtout distrait par les malheurs de ma patrie et les miens propres," he still had the vigor of spirit and the leisure to conceive the great work which he published (1822) eight years later. In this work was first made prominent the power of central pro jection in demonstration and the power of the principle of continuity in research. His leading idea was the study of Projective properties, and as a foundation principle he introduced the anharmonic ratio, a concept, however, which dates back to Pappos and which Desargues (1639) had also used. Möbius, following Poncelet, made much use of the anharmonic ratio in his Barycentrische Calcül (1827), but under the name "Doppelschnitt-Verhältniss" (ratio bisectionalis), a term now in common use under Steiner's abbreviated form "Doppelverhältniss." The name "anharmonic ratio" or "function" (rapport anharmonique, or fonction anharmonique) is due to Chasles, and "cross-ratio" was coined by Clifford. The anharmonic point and line properties of conics have been further elaborated by Brianchon, Chasles, Steiner, and von Staudt. To Poncelet is also due the theory of "figures homologiques," the perspective axis and perspective center (called by Chasles the axis and center of homology), an extension of Carnot's theory of transversals, and the "cordes idéales" of conics which Plücker applied to curves of all orders, He also discovered what Salmon has called "the circular points at infinity," thus completing and establishing the first great principle of modern geometry, the principle of continuity. Brianchon (1806), through his application of Desargues's theory of polars, completed the foundation which Monge had begun for Poncelet's (1829) theory of reciprocal polars.

Among the most prominent geometers contemporary with Poncelet was Gergonne, who with more propriety might be ranked as an analytic geometer. He first (1813) used the term "polar" in its modern geometric sense, although Servois (1811) had used the expression "pole." He was also the first (1825-26) to grasp the idea that the parallelism which Maurolycus, Snell, and Viete had noticed is a fundamental principle. This principle he stated and to it he gave the name which it now bears, the Principle of

Duality, the most important, after that of continuity, in modern geometry. This principle of geometric reciprocation, the discovery of which was also claimed by Poncelet, has been greatly elaborated and has found its way into modern algebra and elementary geometry, and has recently been extended to mechanics by Genese. Gergonne was the first to use the word "class" in describing a curve, explicitly defining class and degree (order) and showing the duality between the two. He and Chasles were among the first to study scientifically surfaces of higher order.

Steiner (1832) gave the first complete discussion of the Projective relations between rows, pencils, etc., and laid the foundation for the subsequent development of pure geometry. He practically closed the theory of conic sections, of the corresponding figures in three-dimensional space and of surfaces of the second order, and hence with him opens the period of special study of curves and surfaces of higher order. His treatment of duality and his application of the theory of Projective pencils to the generation of conics are masterpieces. The theory of polars of a point in regard to a curve had been studied by Bobillier and by Grassmann, but Steiner (1848) showed that this theory can serve as the foundation for the study of plane curves independently of the use of coordinates, and introduced those noteworthy curves covariant to a given curve which now bear the names of himself, Hesse, and Cayley. This whole subject has been extended by Grassmann, Chasles, Cremona, and Jonquières. Steiner was the first to make prominent (1832) an example of correspondence of a more complicated nature than that of Poncelet, Möbius, Magnus, and Chasles. His contributions, and those of Gudermann, to the geometry of the sphere were also noteworthy.

While Möbius, Plücker, and Steiner were at work in Germany, Chasles was closing the geometric era opened in France by Monge. His Aperçu Historique (1837) is a classic, and did for France what Salmon's works did for algebra and geometry in England, popularizing the researches of earlier writers and contributing both to the theory and the nomenclature of the subject. To him is due the name "homographic" and the complete exposition of the principle as applied to plane and solid figures, a subject which has received attention in England at the hands of Salmon, Townsend, and H. J. S. Smith.

Von Staudt began his labors after Plücker, Steiner, and Chasles had made their greatest contributions, but in spite of this seeming disadvantage he surpassed them all. Joining the Steiner school, as opposed to that of Plücker, he became the greatest exponent of pure synthetic geometry of modern times. He set forth (1847, 1856-60) a complete, pure geometric system in which metrical geometry finds no place. Projective properties foreign to measurements are established independently of number relations, number being drawn from geometry instead of conversely, and imaginary elements being systematically introduced from the geometric side. A Projective geometry based on the group containing all the real Projective and dualistic transformations, is developed, imaginary transformations being also introduced. Largely through his influence pure geometry again became a fruitful field. Since his time the distinction between the metrical and Projective theories has been to a great extent obliterated,3 the metrical properties being considered as Projective relations to a fundamental configuration, the circle at infinity common for all spheres. Unfortunately von Staudt wrote in an unattractive style, and to Reye is due much of the popularity which now attends the subject.

Cremona began his publications in 1862. His elementary work on Projective geometry (1875) in Leudesdorf 's translation is familiar to English readers. His contributions to the theory of geometric transformations are valuable, as also his works on plane curves, surfaces, etc.

In England Mulcahy, but especially Townsend (1863), and Hirst, a pupil of Steiner's, opened the subject of modern geometry. Clifford did much to make known the German theories, besides himself contributing to the study of polars and the general theory of curves.

1 Wiener, Chr., Lehrbuch der darstellenden Geometrie, Leipzig, 1884-87; Geschichte der darstellenden Geometrie, 1884.

2 On recent development of graphic methods and the influence of Monge upon this branch of mathematics, see Eddy, H. T., Modern Graphical Developments, Mathematical Papers of Chicago Congress (New York, 1896), p 58.

3 Klein, F., Erlangen Programme of 1872, Haskell's translation, Bulletin of New York Mathematical Society, Vol. II, p. 215.