History of Modern Mathematics

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ARTICLE 6: THEORY OF EQUATIONS

The Theory of Numerical Equations1 concerns itself first with the location of the roots, and then with their approximation. Neither problem is new, but the first noteworthy contribution to the former in the nineteenth century was Budan's (1807). Fourier's work was undertaken at about the same time, but appeared posthumously in 1831. All processes were, however, exceedingly cumbersome until Sturm (1829) communicated to the French Academy the famous theorem which bears his name and which constitutes one of the most brilliant discoveries of algebraic analysis.

The Approximation of the Roots, once they are located, can be made by several processes. Newton (1711), for example, gave a method which Fourier perfected; and Lagrange (1767) discovered an ingenious way of expressing the root as a continued fraction, a process which Vincent (1836) elaborated. It was, however, reserved for Horner (1819) to suggest the most practical method yet known, the one now commonly used. With Horner and Sturm this branch practically closes. The calculation of the imaginary roots by approximation is still an open field.

The Fundamental Theorem2 that every numerical equation has a root was generally assumed until the latter part of the eighteenth century. D'Alembert (1746) gave a demonstration, as did Lagrange (1772), Laplace (1795), Gauss (1799) and Argand (1806). The general theorem that every algebraic equation of the nth degree has exactly n roots and no more follows as a special case of Cauchy's proposition (1831) as to the number of roots within a given contour. Proofs are also due to Gauss, Serret, Clifford (1876), Malet (1878), and many others.

The Impossibility of Expressing the Roots of an equation as algebraic functions of the coefficients when the degree exceeds 4 was anticipated by Gauss and announced by Ruffini, and the belief in the fact became strengthened by the failure of Lagrange's methods for these cases. But the first strict proof is due to Abel, whose early death cut short his labors in this and other fields.

The Quintic Equation has naturally been an object of special study. Lagrange showed that its solution depends on that of a sextic, "Lagrange's resolvent sextic," and Malfatti and Vandermonde investigated the construction of resolvents. The resolvent sextic was somewhat simplified by Cockle and Harley (1858-59) and by Cayley (1861), but Kronecker (1858) was the first to establish a resolvent by which a real simplification was effected. The transformation of the general quintic into the trinomial form x5 + ax + b = 0 by the extraction of square and cube roots only, was first shown to be possible by Bring (1786) and independently by Jerrard3 (1834). Hermite (1858) actually effected this reduction, by means of Tschirnhausen's theorem, in connection with his solution by elliptic functions.

The Modern Theory of Equations may be said to date from Abel and Galois. The latter's special memoir on the subject, not published until 1846, fifteen years after his death, placed the theory on a definite base. To him is due the discovery that to each equation corresponds a group of substitutions (the "group of the equation") in which are reflected its essential characteristics.4 Galois's untimely death left without sufficient demonstration several important propositions, a gap which Betti (1852) has filled. Jordan, Hermite, and Kronecker were also among the earlier ones to add to the theory. Just prior to Galois's researches Abel (1824), proceeding from the fact that a rational function of five letters having less than five values cannot have more than two, showed that the roots of a general quintic equation cannot be expressed in terms of its coefficients by means of radicals. He then investigated special forms of quintic equations which admit of solution by the extraction of a finite number of roots. Hermite, Sylvester, and Brioschi have applied the invariant theory of binary forms to the same subject.

From the point of view of the group the solution by radicals, formerly the goal of the algebraist, now appears as a single link in a long chain of questions relative to the transformation of irrationals and to their classification. Klein (1884) has handled the whole subject of the quintic equation in a simple manner by introducing the icosahedron equation as the normal form, and has shown that the method can be generalized so as to embrace the whole theory of higher equations.5 He and Gordan (from 1879) have attacked those equations of the sixth and seventh degrees which have a Galois group of 168 substitutions, Gordan performing the reduction of the equation of the seventh degree to the ternary problem. Klein (1888) has shown that the equation of the twenty-seventh degree occurring in the theory of cubic surfaces can be reduced to a normal problem in four variables, and Burkhardt (1893) has performed the reduction, the quaternary groups involved having been discussed by Maschke (from 1887).

Thus the attempt to solve the quintic equation by means of radicals has given place to their treatment by transcendents. Hermite (1858) has shown the possibility of the solution, by the use of elliptic functions, of any Bring quintic, and hence of any equation of the fifth degree. Kronecker (1858), working from a different standpoint, has reached the same results, and his method has since been simplified by Brioschi. More recently Kronecker, Gordan, Kiepert, and Klein, have contributed to the same subject, and the sextic equation has been attacked by Maschke and Brioschi through the medium of hyperelliptic functions.

Binomial Equations, reducible to the form xn – 1 = 0, admit of ready solution by the familiar trigonometric formula x = cos 2kπ/n + i sin 2kπ/n; but it was reserved for Gauss (1801) to show that an algebraic solution is possible. Lagrange (1808) extended the theory, and its application to geometry is one of the leading additions of the century. Abel, generalizing Gauss's results, contributed the important theorem that if two roots of an irreducible equation are so connected that the one can be expressed rationally in terms of the other, the equation yields to radicals if the degree is prime and otherwise depends on the solution of lower equations. The binomial equation, or rather the equation , is one of this class considered by Abel, and hence called (by Kronecker) Abelian Equations. The binomial equation has been treated notably by Richelot (1832), Jacobi (1837), Eisenstein (1844, 1850), Cayley (1851), and Kronecker (1854), and is the subject of a treatise by Bachmann (1872). Among the most recent writers on Abelian equations is Pellet (1891).

Certain special equations of importance in geometry have been the subject of study by Hesse, Steiner, Cayley, Clebsch, Salmon, and Kummer. Such are equations of the ninth degree determining the points of inflection of a curve of the third degree, and of the twenty-seventh degree determining the points in which a curve of the third degree can have contact of the fifth order with a conic.

Symmetric Functions of the coefficients, and those which remain unchanged through some or all of the permutations of the roots, are subjects of great importance in the present theory. The first formulas for the computation of the symmetric functions of the roots of an equation seem to have been worked out by Newton, although Girard (1629) had given, without proof, a formula for the power sum. In the eighteenth century Lagrange (1768) and Waring (1770, 1782) contributed to the theory, but the first tables, reaching to the tenth degree, appeared in 1809 in the Meyer-Hirsch Aufgabensammlung. In Cauchy's celebrated memoir on determinants (1812) the subject began to assume new prominence, and both he and Gauss (1816) made numerous and valuable contributions to the theory. It is, however, since the discoveries by Galois that the subject has become one of great importance. Cayley (1857) has given simple rules for the degree and weight of symmetric functions, and he and Brioschi have simplified the computation of tables.

Methods of Elimination and of finding the resultant (Bezout) or eliminant (De Morgan) occupied a number of eighteenth-century algebraists, prominent among them being Euler (1748), whose method, based on symmetric functions, was improved by Cramer (1750) and Bezout (1764). The leading steps in the development are represented by Lagrange (1770-71), Jacobi, Sylvester (1840), Cayley (1848, 1857), Hesse (1843, 1859), Bruno (1859), and Katter (1876). Sylvester's dialytic method appeared in 1841, and to him is also due (1851) the name and a portion of the theory of the discriminant. Among recent writers on the general theory may be mentioned Burnside and Pellet (from 1887).

1 Cayley, A., Equations, and Kelland, P., Algebra, in Encyclopædia Britannica; Favaro, A., Notizie storico-critiche sulla costruzione delle equazioni. Modena, 1878; Cantor, M., Geschichte der Mathematik, Vol. III, p. 375.

2 Loria, Gino, Esame di alcune ricerche concernenti l'esistenza di radici nelle equazioni algebriche; Bibliotheca Mathematica, 1891, p. 99; bibliography on p. 107. Pierpont, J., On the Ruffini-Abelian theorem, Bulletin of American Mathematical Society, Vol. II, p. 200.

3 Harley, R., A contribution of the history . . . of the general equation of the fifth degree, Quarterly Journal of Mathematics, Vol. VI, p. 38.

4 See Art. 7.

5 Klein, F., Vorlesungen über das Ikosaeder, 1884.