# History of Modern Mathematics

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## ARTICLE 7: SUBSTITUTIONS AND GROUPS

The Theories of Substitutions and Groups1 are among the most important in the whole mathematical field, the study of groups and the search for invariants now occupying the attention of many mathematicians. The first recognition of the importance of the combinatory analysis occurs in the problem of forming an mth-degree equation having for roots m of the roots of a given nth-degree equation (m < n). For simple cases the problem goes back to Hudde (1659). Saunderson (1740) noted that the determination of the quadratic factors of a biquadratic expression necessarily leads to a sextic equation, and Le Sœur (1748) and Waring (1762 to 1782) still further elaborated the idea.

Lagrange2 first undertook a scientific treatment of the theory of substitutions. Prior to his time the various methods of solving lower equations had existed rather as isolated artifices than as unified theory.3 Through the great power of analysis possessed by Lagrange (1770, 1771) a common foundation was discovered, and on this was built the theory of substitutions. He undertook to examine the methods then known, and to show a priori why these succeeded below the quintic, but otherwise failed. In his investigation he discovered the important fact that the roots of all resolvents (résolvantes, réduites) which he examined are rational functions of the roots of the respective equations. To study the properties of these functions he invented a "Calcul des Combinaisons." the first important step towards a theory of substitutions. Mention should also be made of the contemporary labors of Vandermonde (1770) as foreshadowing the coming theory.

The next great step was taken by Ruffini4 (1799). Beginning like Lagrange with a discussion of the methods of solving lower equations, he attempted the proof of the impossibility of solving the quintic and higher equations. While the attempt failed, it is noteworthy in that it opens with the classification of the various "permutations" of the coefficients, using the word to mean what Cauchy calls a "système des substitutions conjuguées," or simply a "système conjugué," and Galois calls a "group of substitutions." Ruffini distinguishes what are now called intransitive, transitive and imprimitive, and transitive and primitive groups, and (1801) freely uses the group of an equation under the name "l'assieme della permutazioni." He also publishes a letter from Abbati to himself, in which the group idea is prominent.

To Galois, however, the honor of establishing the theory of groups is generally awarded. He found that if r1, r2, . . . rn are the n roots of an equation, there is always a group of permutations of the r's such that (1) every function of the roots invariable by the substitutions of the group is rationally known, and (2), reciprocally, every rationally determinable function of the roots is invariable by the substitutions of the group. Galois also contributed to the theory of modular equations and to that of elliptic functions. His first publication on the group theory was made at the age of eighteen (1829), but his contributions attracted little attention until the publication of his collected papers in 1846 (Liouville, Vol. XI).

Cayley and Cauchy were among the first to appreciate the importance of the theory, and to the latter especially are due a number of important theorems. The popularizing of the subject is largely due to Serret, who has devoted section IV of his algebra to the theory; to Camille Jordan, whose Traité des Substitutions is a classic; and to Netto (1882), whose work has been translated into English by Cole (1892). Bertrand, Hermite, Frobenius, Kronecker, and Mathieu have added to the theory. The general problem to determine the number of groups of n given letters still awaits solution.

But overshadowing all others in recent years in carrying on the labors of Galois and his followers in the study of discontinuous groups stand Klein, Lie, Poincaré, and Picard. Besides these discontinuous groups there are other classes, one of which, that of finite continuous groups, is especially important in the theory of differential equations. It is this class which Lie (from 1884) has studied, creating the most important of the recent departments of mathematics, the theory of transformation groups. Of value, too, have been the labors of Killing on the structure of groups, Study's application of the group theory to complex numbers, and the work of Schur and Maurer.

1 Netto, E., Theory of Substitutions, translated by Cole; Cayley, A., Equations, Encyclopædia Britannica, 9th edition.

2 Pierpont, James, Lagrange's Place in the Theory of Substitutions, Bulletin of American Mathematical Society, Vol. I, p. 196.

3 Matthiessen, L. Grundzüge der antiken und modernen Algebra der litteralen Gleichungen, Leipzig, 1878.

4 Burkhardt, H., Die Anfänge der Gruppentheorie und Paolo Ruffini, Abhandlungen zur Geschichte der Mathematik, VI, 1892, p. 119. Italian by E. Pascal, Brioschi's Annali di Matematica, 1894.