ARTICLE 9: QUANTICS

The Theory of Qualities or Forms1 appeared in embryo in the Berlin memoirs of Lagrange (1773, 1775), who considered binary quadratic forms of the type ax2 + bxy + cy2, and established the invariance of the discriminant of that type when x + □y is put for x. He classified forms of that type according to the sign of b2 – 4ac, and introduced the ideas of transformation and equivalence. Gauss2 (1801) next took up the subject, proved the invariance of the discriminants of binary and ternary quadratic forms, and systematized the theory of binary quadratic forms, a subject elaborated by H. J. S. Smith, Eisenstein, Dirichlet, Lipschitz, Poincaré, and Cayley. Galois also entered the field, in his theory of groups (1829), and the first step towards the establishment of the distinct theory is sometimes attributed to Hesse in his investigations of the plane curve of the third order.

It is, however, to Boole (1841) that the real foundation of the theory of invariants is generally ascribed. He first showed the generality of the invariant property of the discriminant, which Lagrange and Gauss had found for special forms. Inspired by Boole's discovery Cayley took up the study in a memoir "On the Theory of Linear Transformations" (1845), which was followed (1846) by investigations concerning covariants and by the discovery of the symbolic method of finding invariants. By reason of these discoveries concerning invariants and covariants (which at first he called "hyperdeterminants") he is regarded as the founder of what is variously called Modern Algebra, Theory of Forms, Theory of Quantics, and the Theory of Invariants and Covariants. His ten memoirs on the subject began in 1854, and rank among the greatest which have ever been produced upon a single theory. Sylvester soon joined Cayley in this work, and his originality and vigor in discovery soon made both himself and the subject prominent. To him are due (1851-54) the foundations of the general theory, upon which later writers have largely built, as well as most of the terminology of the subject.

Meanwhile in Germany Eisenstein (1843) had become aware of the simplest invariants and covariants of a cubic and biquadratic form, and Hesse and Grassmann had both (1844) touched upon the subject. But it was Aronhold (1849) who first made the new theory known. He devised the symbolic method now common in Germany, discovered the invariants of a ternary cubic and their relations to the discriminant, and, with Cayley and Sylvester, studied those differential equations which are satisfied by invariants and covariants of binary quantics. His symbolic method has been carried on by Clebsch, Gordan, and more recently by Study (1889) and Stroh (1890), in lines quite different from those of the English school.

In France Hermite early took up the work (1851). He discovered (1854) the law of reciprocity that to every covariant or invariant of degree □ and order r of a form of the mth order corresponds also a covariant or invariant of degree m and of order r of a form of the □th order. At the same time (1854) Brioschi joined the movement, and his contributions have been among the most valuable. Salmon's Higher Plane Curves (1852) and Higher Algebra (1859) should also be mentioned as marking an epoch in the theory.

Gordan entered the field, as a critic of Cayley, in 1868. He added greatly to the theory, especially by his theorem on the Endlichkeit des Formensystems, the proof for which has since been simplified. This theory of the finiteness of the number of invariants and covariants of a binary form has since been extended by Peano (1882), Hilbert (1884), and Mertens (1886). Hilbert (1890) succeeded in showing the finiteness of the complete systems for forms in n variables, a proof which Story has simplified.

Clebsch3 did more than any other to introduce into Germany the work of Cayley and Sylvester, interpreting the projective geometry by their theory of invariants, and correlating it with Riemann's theory of functions. Especially since the publication of his work on forms (1871) the subject has attracted such scholars as Weierstrass, Kronecker, Mansion, Noether, Hilbert, Klein, Lie, Beltrami, Burkhardt, and many others. On binary forms Faà di Bruno's work is well known, as is Study's (1889) on ternary forms. De Toledo (1889) and Elliott (1895) have published treatises on the subject.

Dublin University has also furnished a considerable corps of contributors, among whom MacCullagh, Hamilton, Salmon, Michael and Ralph Roberts, and Burnside may be especially mentioned. Burnside, who wrote the latter part of Burnside and Panton's Theory of Equations, has set forth a method of transformation which is fertile in geometric interpretation and binds together binary and certain ternary forms.

The equivalence problem of quadratic and bilinear forms has attracted the attention of Weierstrass, Kronecker, Christoffel, Frobenius, Lie, and more recently of Rosenow (Crelle, 108), Werner (1889), Killing (1890), and Scheffers (1891). The equivalence problem of non-quadratic forms has been studied by Christoffel. Schwarz

(1872), Fuchs (1875-76), Klein (1877, 1884), Brioschi (1877), and Maschke (1887) have contributed to the theory of forms with linear transformations into themselves. Cayley (especially from 1870) and Sylvester (1877) have worked out the methods of denumeration by means of generating functions. Differential invariants have been studied by Sylvester, MacMahon, and Hammond. Starting from the differential invariant, which Cayley has termed the Schwarzian derivative, Sylvester (1885) has founded the theory of reciprocants, to which MacMahon, Hammond, Leudesdorf, Elliott, Forsyth, and Halphen have contributed. Canonical forms have been studied by Sylvester (1851), Cayley, and Hermite (to whom the term "canonical form" is due), and more recently by Rosanes (1873), Brill (1882), Gundelfinger (1883), and Hilbert (1886).

The Geometric Theory of Binary Forms may be traced to Poncelet and his followers. But the modern treatment has its origin in connection with the theory of elliptic modular functions, and dates from Dedekind's letter to Borchardt (Crelle, 1877). The names of Klein and Hurwitz are prominent in this connection. On the method of nets (réseaux), another geometric treatment of binary quadratic forms Gauss (1831), Dirichlet (1850), and Poincaré (1880) have written.

1 Meyer, W. F., Bericht über den gegenwärtigen Stand der Invariantentheorie. Jahresbericht der deutschen Mathematiker-Vereinigung, Vol. I, 1890-91; Berlin 1892, p. 97. See also the review by Franklin in Bulletin New York Mathematical Society, Vol. III, p. 187; Biography of Cayley, Collected Papers, VIII, p. ix, and Proceedings of Royal Society, 1895.

2 See Art. 2.

3 Klein's Evanston Lectures, Lect. I.