History of Modern Mathematics

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Article 11: Differential

The Theory of Differential Equations[1] has been called by Lie[2] the most important of modern mathematics. The influence of geometry, physics, and astronomy, starting with Newton and Leibniz, and further manifested through the Bernoullis, Riccati, and Clairaut, but chiefly through d'Alembert and Euler, has been very marked, and especially on the theory of linear partial differential equations with constant coefficients. The first method of integrating linear ordinary differential equations with constant coefficients is due to Euler, who made the solution of his type, depend on that of the algebraic equation of the nth degree, $F(z) = z^n + A_1 z^{n-1} + \cdots + A_n = 0$, in which z^k takes the place of . This equation $F(z) = 0$, is the "characteristic" equation considered later by Monge and Cauchy.

The theory of linear partial differential equations may be said to begin with Lagrange (1779 to 1785). Monge (1809) treated ordinary and partial differential equations of the first and second order, uniting the theory to geometry, and introducing the notion of the "characteristic," the curve represented by $F(z) = 0$, which has recently been investigated by Darboux, Levy, and Lie. Pfaff (1814, 1815) gave the first general method of integrating partial differential equations of the first order, a method of which Gauss (1815) at once recognized the value and of which he gave an analysis. Soon after, Cauchy (1819) gave a simpler method, attacking the subject from the analytical standpoint, but using the Monge characteristic. To him is also due the theorem, corresponding to the fundamental theorem of algebra, that every differential equation defines a function expressible by means of a convergent series, a proposition more simply proved by Briot and Bouquet, and also by Picard (1891). Jacobi (1827) also gave an analysis of Pfaff 's method, besides developing an original one (1836) which Clebsch published (1862). Clebsch's own method appeared in 1866, and others are due to Boole (1859), Korkine (1869), and A. Mayer (1872). Pfaff 's problem has been a prominent subject of investigation, and with it are connected the names of Natani (1859), Clebsch (1861, 1862), DuBois-Reymond (1869), Cayley, Baltzer, Frobenius, Morera, Darboux, and Lie. The next great improvement in the theory of partial differential equations of the first order is due to Lie (1872), by whom the whole subject has been placed on a rigid foundation. Since about 1870, Darboux, Kovalevsky, Méray, Mansion, Graindorge, and Imschenetsky have been prominent in this line. The theory of partial differential equations of the second and higher orders, beginning with Laplace and Monge, was notably advanced by Ampère (1840). Imschenetsky3 has summarized the contributions to 1873, but the theory remains in an imperfect state.

The integration of partial differential equations with three or more variables was the object of elaborate investigations by Lagrange, and his name is still connected with certain subsidiary equations. To him and to Charpit, who did much to develop the theory, is due one of the methods for integrating the general equation with two variables, a method which now bears Charpit's name.

The theory of singular solutions of ordinary and partial differential equations has been a subject of research from the time of Leibniz, but only since the middle of the present century has it received especial attention. A valuable but little-known work on the subject is that of Houtain (1854). Darboux (from 1873) has been a leader in the theory, and in the geometric interpretation of these solutions he has opened a field which has been worked by various writers, notably Casorati and Cayley. To the latter is due (1872) the theory of singular solutions of differential equations of the first order as at present accepted.

The primitive attempt in dealing with differential equations had in view a reduction to quadratures. As it had been the hope of eighteenth century algebraists to find a method for solving the general equation of the nth degree, so it was the hope of analysts to find a general method for integrating any differential equation. Gauss (1799) showed, however, that the differential equation meets its limitations very soon unless complex numbers are introduced. Hence analysts began to substitute the study of functions, thus opening a new and fertile field. Cauchy was the first to appreciate the importance of this view, and the modern theory may be said to begin with him. Thereafter the real question was to be, not whether a solution is possible by means of known functions or their integrals, but whether a given differential equation suffices for the definition of a function of the independent variable or variables, and if so, what are the characteristic properties of this function.

Within a half-century the theory of ordinary differential equations has come to be one of the most important branches of analysis, the theory of partial differential equations remaining as one still to be perfected. The difficulties of the general problem of integration are so manifest that all classes of investigators have confined themselves to the properties of the integrals in the neighborhood of certain given points. The new departure took its greatest inspiration from two memoirs by Fuchs (Crelle, 1866, 1868), a work elaborated by Thomé and Frobenius. Collet has been a prominent contributor since 1869, although his method for integrating a non-linear system was communicated to Bertrand in 1868. Clebsch4 (1873) attacked the theory along lines parallel to those followed in his theory of Abelian integrals. As the latter can be classified according to the properties of the fundamental curve which remains unchanged under a rational transformation, so Clebsch proposed to classify the transcendent functions defined by the differential equations according to the invariant properties of the corresponding surfaces f = 0 under rational one-to-one transformations.

Since 1870 Lie's5 labors have put the entire theory of differential equations on a more satisfactory foundation. He has shown that the integration theories of the older mathematicians, which had been looked upon as isolated, can by the introduction of the concept of continuous groups of transformations be referred to a common source, and that ordinary differential equations which admit the same infinitesimal transformations present like difficulties of integration. He has also emphasized the subject of transformations of contact (Berührungstransformationen) which underlies so much of the recent theory. The modern school has also turned its attention to the theory of differential invariants, one of fundamental importance and one which Lie has made prominent. With this theory are associated the names of Cayley, Cockle, Sylvester, Forsyth, Laguerre, and Halphen. Recent writers have shown the same tendency noticeable in the work of Monge and Cauchy, the tendency to separate into two schools, the one inclining to use the geometric diagram, and represented by Schwarz, Klein, and Goursat, the other adhering to pure analysis, of which Weierstrass, Fuchs, and Frobenius are types. The work of Fuchs and the theory of elementary divisors have formed the basis of a late work by Sauvage (1895). Poincaré's recent contributions are also very notable. His theory of Fuchsian equations (also investigated by Klein) is connected with the general theory. He has also brought the whole subject into close relations with the theory of functions. Appell has recently contributed to the theory of linear differential equations transformable into themselves by change of the function and the variable. Helge von Koch has written on infinite determinants and linear differential equations. Picard has undertaken the generalization of the work of Fuchs and Poincaré in the case of differential equations of the second order. Fabry (1885) has generalized the normal integrals of Thomé, integrals which Poincaré has called "intégrales anormales," and which Picard has recently studied. Riquier has treated the question of the existence of integrals in any differential system and given a brief summary of the history to 1895.6 The number of contributors in recent times is very great, and includes, besides those already mentioned, the names of Brioschi, Königsberger, Peano, Graf, Hamburger, Graindorge, Schläfli, Glaisher, Lommel, Gilbert, Fabry, Craig, and Autonne.

1 Cantor, M., Geschichte der Mathematik, Vol. III, p. 429; Schlesinger, L., Handbuch der Theorie der linearen Differentialgleichungen, Vol. I, 1895, an excellent historical view; review by Mathews in Nature, Vol. LII, p. 313; Lie, S., Zur allgemeinen Theorie der partiellen Differentialgleichungen, Berichte über die Verhandlungen der Gesellschaft der Wissenschaften zu Leipzig, 1895; Mansion, P., Theorie der partiellen Differentialgleichungen ter Ordnung, German by Maser, Leipzig, 1892, excellent on history; Craig, T., Some of the Developments in the Theory of Ordinary Differential Equations, 1878-1893, Bulletin New York Mathematical Society, Vol. II, p. 119 ; Goursat, E., Leçons sur l'intégration des équations aux dérivées partielles du premier ordre, Paris, 1895; Burkhardt, H., and Heffier, L., in Mathematical Papers of Chicago Congress, p.13 and p. 96.

2 "In der ganzen modernen Mathematik ist die Theorie der Differentialgleichungen die wichtigste Disciplin."

3 Grunert's Archiv für Mathematik, Vol. LIV.

4 Klein's Evanston Lectures, Lect. I. 5 Klein's Evanston Lectures, Lect. II, III.

5 Klein's Evanston Lectures, Lect. II, III.

6 Riquier, C., Mèmoire sur l'existence des intègrales dans un système differentiel quelconque, etc. Mèmoires des Savants ètrangers, Vol. XXXII, No. 3.