History of Modern Mathematics

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ARTICLE 13: THEORY OF FUNCTION

The Theory of Functions1 may be said to have its first development in Newton's works, although algebraists had already become familiar with irrational functions in considering cubic and quartic equations. Newton seems first to have grasped the idea of such expressions in his consideration of symmetric functions of the roots of an equation. The word was employed by Leibniz (1694), but in connection with the Cartesian geometry. In its modern sense it seems to have been first used by Johann Bernoulli, who distinguished between algebraic and transcendent functions. He also used (1718) the function symbol □. Clairaut (1734) used □x, □x, Δx, for various functions of x, a symbolism substantially followed by d'Alembert (1747) and Euler (1753). Lagrange (1772, 1797, 1806) laid the foundations for the general theory, giving to the symbol a broader meaning, and to the symbols f, □, F, · · ·, f □, □ □, F □, · · · their modern signification. Gauss contributed to the theory, especially in his proofs of the fundamental theorem of algebra, and discussed and gave name to the theory of "conforme Abbildung," the "orthomorphosis" of Cayley.

Making Lagrange's work a point of departure, Cauchy so greatly developed the theory that he is justly considered one of its founders. His memoirs extend over the period 1814-1851, and cover subjects like those of integrals with imaginary limits, infinite series and questions of convergence, the application of the infinitesimal calculus to the theory of complex numbers, the investigation of the fundamental laws of mathematics, and numerous other lines which appear in the general theory of functions as considered to-day. Originally opposed to the movement started by Gauss, the free use of complex numbers, he finally became, like Abel, its advocate. To him is largely due the present orientation of mathematical research, making prominent the theory of functions, distinguishing between classes of functions, and placing the whole subject upon a rigid foundation. The historical development of the general theory now becomes so interwoven with that of special classes of functions, and notably the elliptic and Abelian, that economy of space requires their treatment together, and hence a digression at this point.

The Theory of Elliptic Functions2 is usually referred for its origin to Landen's (1775) substitution of two elliptic arcs for a single hyperbolic arc. But Jakob Bernoulli (1691) had suggested the idea of comparing non-congruent arcs of the same curve, and Johann had followed up the investigation. Fagnano (1716) had made similar studies, and both Maclaurin (1742) and d'Alembert (1746) had come upon the borderland of elliptic functions. Euler (from 1761) had summarized and extended the rudimentary theory, showing the necessity for a convenient notation for elliptic arcs, and prophesying (1766) that "such signs will afford a new sort of calculus of which I have here attempted the exposition of the first elements." Euler's investigations continued until about the time of his death (1783), and to him Legendre attributes the foundation of the theory. Euler was probably never aware of Landen's discovery.

It is to Legendre, however, that the theory of elliptic functions is largely due, and on it his fame to a considerable degree depends. His earlier treatment (1786) almost entirely substitutes a strict analytic for the geometric method. For forty years he had the theory in hand, his labor culminating in his Traité des Fonctions elliptiques et des Intégrales Eulériennes (1825-28). A surprise now awaiting him is best told in his own words: "Hardly had my work seen the light–its name could scarcely have become known to scientific foreigners,–when I learned with equal surprise and satisfaction that two young mathematicians, MM. Jacobi of Königsberg and Abel of Christiania, had succeeded by their own studies in perfecting considerably the theory of elliptic functions in its highest parts." Abel began his contributions to the theory in 1825, and even then was in possession of his fundamental theorem which he communicated to the Paris Academy in 1826. This communication being so poorly transcribed was not published in full until 1841, although the theorem was sent to Crelle (1829) just before Abel's early death. Abel discovered the double periodicity of elliptic functions, and with him began the treatment of the elliptic integral as a function of the amplitude.

Jacobi, as also Legendre and Gauss, was especially cordial in praise of the delayed theorem of the youthful Abel. He calls it a "monumentum ære perennius," and his name "das Abel'sche Theorem" has since attached to it. The functions of multiple periodicity to which it refers have been called Abelian Functions. Abel's work was early proved and elucidated by Liouville and Hermite. Serret and Chasles in the Comptes Rendus, Weierstrass (1853), Clebsch and Gordan in their Theorie der Abel'schen Functionen (1866), and Briot and Bouquet in their two treatises have greatly elaborated the theory. Riemann's3 (1857) celebrated memoir in Crelle presented the subject in such a novel form that his treatment was slow of acceptance. He based the theory of Abelian integrals and their inverse, the Abelian functions, on the idea of the surface now so well known by his name, and on the corresponding fundamental existence theorems. Clebsch, starting from an algebraic curve defined by its equation, made the subject more accessible, and generalized the theory of Abelian integrals to a theory of algebraic functions with several variables, thus creating a branch which has been developed by Noether, Picard, and Poincaré. The introduction of the theory of invariants and projective geometry into the domain of hyperelliptic and Abelian functions is an extension of Clebsch's scheme. In this extension, as in the general theory of Abelian functions, Klein has been a leader. With the development of the theory of Abelian functions is connected a long list of names, including those of Schottky, Humbert, C. Neumann, Fricke, Königsberger, Prym, Schwarz, Painlevé, Hurwitz, Brioschi, Borchardt, Cayley, Forsyth, and Rosenhain, besides others already mentioned.

Returning to the theory of elliptic functions, Jacobi (1827) began by adding greatly to Legendre's work. He created a new notation and gave name to the "modular equations" of which he made use. Among those who have written treatises upon the elliptic-function theory are Briot and Bouquet, Laurent, Halphen, Königsberger, Hermite, Durège, and Cayley, The introduction of the subject into the Cambridge Tripos (1873), and the fact that Cayley's only book was devoted to it, have tended to popularize the theory in England.

The Theory of Theta Functions was the simultaneous and independent creation of Jacobi and Abel (1828). Gauss's notes show that he was aware of the properties of the theta functions twenty years earlier, but he never published his investigations. Among the leading contributors to the theory are Rosenhain (1846, published in 1851) and Göpel (1847), who connected the double theta functions with the theory of Abelian functions of two variables and established the theory of hyperelliptic functions in a manner corresponding to the Jacobian theory of elliptic functions. Weierstrass has also developed the theory of theta functions independently of the form of their space boundaries, researches elaborated by Königsberger (1865) to give the addition theorem. Riemann has completed the investigation of the relation between the theory of the theta and the Abelian functions, and has raised theta functions to their present position by making them an essential part of his theory of Abelian integrals. H. J. S. Smith has included among his contributions to this subject the theory of omega functions. Among the recent contributors are Krazer and Prym (1892), and Wirtinger (1895).

Cayley was a prominent contributor to the theory of periodic functions. His memoir (1845) on doubly periodic functions extended Abel's investigations on doubly infinite products. Euler had given singly infinite products for sin x, cos x, and Abel had generalized these, obtaining for the elementary doubly periodic functions expressions for snx, cnx, dnx. Starting from these expressions of Abel's Cayley laid a complete foundation for his theory of elliptic functions. Eisenstein (1847) followed, giving a discussion from the standpoint of pure analysis, of a general doubly infinite product, and his labors, as supplemented by Weierstrass, are classic.

The General Theory of Functions has received its present form largely from the works of Cauchy, Riemann, and Weierstrass. Endeavoring to subject all natural laws to interpretation by mathematical formulas, Riemann borrowed his methods from the theory of the potential, and found his inspiration in the contemplation of mathematics from the standpoint of the concrete. Weierstrass, on the other hand, proceeded from the purely analytic point of view. To Riemann4 is due the idea of making certain partial differential equations, which express the fundamental properties of all functions, the foundation of a general analytical theory, and of seeking criteria for the determination of an analytic function by its discontinuities and boundary conditions. His theory has been elaborated by Klein (1882, and frequent memoirs) who has materially extended the theory of Riemann's surfaces. Clebsch, Lüroth, and later writers have based on this theory their researches on loops. Riemann's speculations were not without weak points, and these have been fortified in connection with the theory of the potential by C. Neumann, and from the analytic standpoint by Schwarz.

In both the theory of general and of elliptic and other functions, Clebsch was prominent. He introduced the systematic consideration of algebraic curves of deficiency 1, bringing to bear on the theory of elliptic functions the ideas of modern pro jective geometry. This theory Klein has generalized in his Theorie der elliptischen Modulfunctionen, and has extended the method to the theory of hyperelliptic and Abelian functions.

Following Riemann came the equally fundamental and original and more rigorously worked out theory of Weierstrass. His early lectures on functions are justly considered a landmark in modern mathematical development. In particular, his researches on Abelian transcendents are perhaps the most important since those of Abel and Jacobi. His contributions to the theory of elliptic functions, including the introduction of the function $\wp(u)$, are also of great importance. His contributions to the general function theory include much of the symbolism and nomenclature, and many theorems. He first announced (1866) the existence of natural limits for analytic functions, a subject further investigated by Schwarz, Klein, and Fricke. He developed the theory of functions of complex variables from its foundations, and his contributions to the theory of functions of real variables were no less marked.

Fuchs has been a prominent contributor, in particular (1872) on the general form of a function with essential singularities. On functions with an infinite number of essential singularities Mittag-Leffler (from 1882) has written and contributed a fundamental theorem. On the classification of singularities of functions Guichard (1883) has summarized and extended the researches, and Mittag-Leffler and G. Cantor have contributed to the same result. Laguerre (from 1882) was the first to discuss the "class" of transcendent functions, a subject to which Poincaré, Cesaro, Vivanti, and Hermite have also contributed. Automorphic functions, as named by Klein, have been investigated chiefly by Poincaré, who has established their general classification. The contributors to the theory include Schwarz, Fuchs, Cayley, Weber, Schlesinger, and Burnside.

The Theory of Elliptic Modular Functions, proceeding from Eisenstein's memoir (1847) and the lectures of Weierstrass on elliptic functions, has of late assumed prominence through the influence of the Klein school. Schläfli (1870), and later Klein, Dyck, Gierster, and Hurwitz, have worked out the theory which Klein and Fricke have embodied in the recent Vorlesungen über die Theorie der elliptischen Modulfunctionen (1890-92). In this theory the memoirs of Dedekind (1877), Klein (1878), and Poincaré (from 1881) have been among the most prominent.

For the names of the leading contributors to the general and special theories, including among others Jordan, Hermite, Hölder, Picard, Biermann, Darboux, Pellet, Reichardt, Burkhardt, Krause, and Humbert, reference must be had to the Brill-Noether Bericht.

Of the various special algebraic functions space allows mention of but one class, that bearing Bessel's name. Bessel's functions5 of the zero order arefound in memoirs of Daniel Bernoulli (1732) and Euler (1764), and before the end of the eighteenth century all the Bessel functions of the first kind and integral order had been used. Their prominence as special functions is due, however, to Bessel (1816-17), who put them in their present form in 1824. Lagrange's series (1770), with Laplace's extension (1777), had been regarded as the best method of solving Kepler's problem (to express the variable quantities in undisturbed planetary motion in terms of the time or mean anomaly), and to improve this method Bessel's functions were first prominently used. Hankel (1869), Lommel (from 1868), F. Neumann, Heine, Graf (1893), Gray and Mathews (1895), and others have contributed to the theory. Lord Rayleigh (1878) has shown the relation between Bessel's and Laplace's functions, but they are nevertheless looked upon as a distinct system of transcendents. Tables of Bessel's functions were prepared by Bessel (1824), by Hansen (1843), and by Meissel (1888).

1 Brill, A., and Noether, M., Die Entwickelung der Theorie der algebraischen Functionen in alterer und neuerer Zeit, Bericht erstattet der Deutschen Mathematiker-Vereinigung, Jahresbericht, Vol. II, pp. 107-566, Berlin, 1894; Königsberger, L., Zur Geschichte der Theorie der elliptischen Transcendenten in den Jahren 1826-29, Leipzig, 1879; Williamson, B., Infinitesimal Calculus, Encyclopædia Britannica; Schlesinger, L., Differentialgleichungen, Vol. I, 1895; Casorati, F., Teorica delle funzioni di variabili complesse, Vol. I, 1868; Klein's Evanston Lectures. For bibliography and historical notes, see Harkness and Morley's Theory of Functions, 1893, and Forsyth's Theory of Functions, 1893; Eneström, G., Note historique sur les symboles . . . Bibliotheca Mathematica, 1891, p. 89.

2 Enneper, A., Elliptische Funktionen, Theorie und Geschichte, Halle, 1890; Königsberger, L., Zur Geschichte der Theorie der elliptischen Transcendenten in den Jahren 1826-29, Leipzig, 1879.

3 Klein, Evanston Lectures, p. 3; Riemann and Modern Mathematics, translated by Ziwet, Bulletin of American Mathematical Society, Vol. I, p. 165; Burkhardt, H., Vortrag über Riemann, Göttingen, 1892.

4 Klein, F., Riemann and Modern Mathematics, translated by Ziwet, Bulletin of American Mathematical Society, Vol. I, p. 165.

5 Bˆocher, M., A bit of mathematical history, Bulletin of New York Mathematical Society, Vol. II, p. 107.