ARTICLE 18: NON-EUCLIDEAN GEOMETRY

The Non-Euclidean Geometry1 is a natural result of the futile attempts which had been made from the time of Proklos to the opening of the nineteenth century to prove the fifth postulate (also called the twelfth axiom, and sometimes the eleventh or thirteenth) of Euclid. The first scientific investigation of this part of the foundation of geometry was made by Saccheri (1733), a work which was not looked upon as a precursor of Lobachevsky, however, until Beltrami (1889) called attention to the fact. Lambert was the next to question the validity of Euclid's postulate, in his Theorie der Parallellinien (posthumous, 1786), the most important of many treatises on the subject between the publication of Saccheri's work and those of Lobachevsky and Bolyai. Legendre also worked in the field, but failed to bring himself to view the matter outside the Euclidean limitations.

During the closing years of the eighteenth century Kant's2 doctrine of absolute space, and his assertion of the necessary postulates of geometry, were the ob ject of much scrutiny and attack. At the same time Gauss was giving attention to the fifth postulate, though on the side of proving it. It was at one time surmised that Gauss was the real founder of the non-Euclidean geometry, his influence being exerted on Lobachevsky through his friend Bartels, and on Johann Bolyai through the father Wolfgang, who was a fellow student of Gauss's. But it is now certain that Gauss can lay no claim to priority of discovery, although the influence of himself and of Kant, in a general way, must have had its effect.

Bartels went to Kasan in 1807, and Lobachevsky was his pupil. The latter's lecture notes show that Bartels never mentioned the subject of the fifth postulate to him, so that his investigations, begun even before 1823, were made on his own motion and his results were wholly original. Early in 1826 he sent forth the principles of his famous doctrine of parallels, based on the assumption that through a given point more than one line can be drawn which shall never meet a given line coplanar with it. The theory was published in full in 1829-30, and he contributed to the subject, as well as to other branches of mathematics, until his death.

Johann Bolyai received through his father, Wolfgang, some of the inspiration to original research which the latter had received from Gauss. When only twenty-one he discovered, at about the same time as Lobachevsky, the principles of non-Euclidean geometry, and refers to them in a letter of November, 1823. They were committed to writing in 1825 and published in 1832. Gauss asserts in his correspondence with Schumacher (1831-32) that he had brought out a theory along the same lines as Lobachevsky and Bolyai, but the publication of their works seems to have put an end to his investigations. Schweikart was also an independent discoverer of the non-Euclidean geometry, as his recently recovered letters show, but he never published anything on the subject, his work on the theory of parallels (1807), like that of his nephew Taurinus (1825), showing no trace of the Lobachevsky-Bolyai idea.

The hypothesis was slowly accepted by the mathematical world. Indeed it was about forty years after its publication that it began to attract any considerable attention. Hoüel (1866) and Flye St. Marie (1871) in France, Riemann (1868), Helmholtz (1868), Frischauf (1872), and Baltzer (1877) in Germany, Beltrami (1872) in Italy, de Tilly (1879) in Belgium, Clifford in England, and Halsted (1878) in America, have been among the most active in making the subject popular. Since 1880 the theory may be said to have become generally understood and accepted as legitimate.3

Of all these contributions the most noteworthy from the scientific standpoint is that of Riemann. In his Habilitationsschrift (1854) he applied the methods of analytic geometry to the theory, and suggested a surface of negative curvature, which Beltrami calls “pseudo-spherical,” thus leaving Euclid’s geometry on a surface of zero curvature midway between his own and Lobachevsky’s. He thus set forth three kinds of geometry, Bolyai having noted only two. These Klein (1871) has called the elliptic (Riemann’s), parabolic (Euclid’s), and hyperbolic (Lobachevsky’s).

Starting from this broader point of view4 there have contributed to the subject many of the leading mathematicians of the last quarter of a century, including, besides those already named, Cayley, Lie, Klein, Newcomb, Pasch, C. S. Peirce, Killing, Fiedler, Mansion, and McClintock. Cayley’s contribution of his “metrical geometry” was not at once seen to be identical with that of Lobachevsky and Bolyai. It remained for Klein (1871) to show this, thus simplifying Cayley’s treatment and adding one of the most important results of the entire theory. Cayley’s metrical formulas are, when the Absolute is real, identical with those of the hyperbolic geometry; when it is imaginary, with the elliptic; the limiting case between the two gives the parabolic (Euclidean) geometry. The question raised by Cayley’s memoir as to how far projective geometry can be defined in terms of space without the introduction of distance had already been discussed by von Staudt (1857) and has since been treated by Klein (1873) and by Lindemann (1876).

1 Stäckel and Engel, Die Theorie der Parallellinien von Euklid bis auf Gauss, Leipzig, 1895; Halsted, G. B., various contributions: Bibliography of Hyperspace and Non-Euclidean Geometry, American Journal of Mathematics, Vols. I, II; The American Mathematical Monthly, Vol. I; translations of Lobachevsky's Geometry, Vasiliev's address on Lobachevsky, Saccheri's Geometry, Bolyai's work and his life; Non-Euclidean and Hyperspaces, Mathematical Papers of Chicago Congress, p. 92. Loria, G., Die hauptsächlichsten Theorien der Geometrie, p. 106; Karagiannides, A., Die Nichteuklidische Geometrie vom Alterthum bis zur Gegenwart, Berlin, 1893; McClintock, E., On the early history of Non-Euclidean Geometry, Bulletin of New York Mathematical Society, Vol. II, p. 144; Poincaré, Non-Euclidean Geom., Nature, 45:404; Articles on Parallels and Measurement in Encyclopædia Britannica, 9th edition; Vasiliev's address (German by Engel) also appears in the Abhandlungen zur Geschichte der Mathematik, 1895.

2 Fink, E., Kant als Mathematiker, Leipzig, 1889.

3 For an excellent summary of the results of the hypothesis, see an article by McClintock, The Non-Euclidian Geometry, Bulletin of New York Mathematical Society, Vol. II, p. 1.

4 Klein. Evanston Lectures. Lect. IX.