Chapter 4

GEORGE BOOLE[1]

(1815-1864)

George Boole was born at Lincoln, England, on the 2d of November, 1815. His father, a tradesman of very limited means, was attached to the pursuit of science, particularly of mathematics, and was skilled in the construction of optical instruments. Boole received his elementary education at the National School of the city, and afterwards at a commercial school; but it was his father who instructed him in the elements of mathematics, and also gave him a taste for the construction and adaptation of optical instruments. However, his early ambition did not urge him to the further prosecution of mathematical studies, but rather to becoming proficient in the ancient classical languages. In this direction he could receive no help from his father, but to a friendly bookseller of the neighborhood he was indebted for instruction in the rudiments of the Latin Grammar. To the study of Latin he soon added that of Greek without any external assistance; and for some years he perused every Greek or Latin author that came within his reach. At the early age of twelve his proficiency in Latin made him the occasion of a literary controversy in his native city. He produced a metrical translation of an ode of Horace, which his father in the pride of his heart inserted in a local journal, stating the age of the translator. A neighboring school-master wrote a letter to the journal in which he denied, from internal evidence, that the version could have been the work of one so young. In his early thirst for knowledge of languages and ambition to excel in verse he was like Hamilton, but poor Boole was much more heavily oppressed by the *res angusta domi*—the hard conditions of his home. Accident discovered to him certain defects in his methods of classical study, inseparable from the want of proper early training, and it cost him two years of incessant labor to correct them.

Between the ages of sixteen and twenty he taught school as an assistant teacher, first at Doncaster in Yorkshire, afterwards at Waddington near Lincoln; and the leisure of these years he devoted mainly to the study of the principal

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modern languages, and of patristic literature with the view of studying to take orders in the Church. This design, however, was not carried out, owing to the financial circumstances of his parents and some other difficulties. In his twentieth year he decided on opening a school on his own account in his native city; thenceforth he devoted all the leisure he could command to the study of the higher mathematics, and solely with the aid of such books as he could procure. Without other assistance or guide he worked his way onward, and it was his own opinion that he had lost five years of educational progress by his imperfect methods of study, and the want of a helping hand to get him over difficulties. No doubt it cost him much time; but when he had finished studying he was already not only learned but an experienced investigator.

We have seen that at this time (1835) the great masters of mathematical analysis wrote in the French language; and Boole was naturally led to the study of the *Mécanique celeste* of Laplace, and the *Mécanique analytique* of Lagrange. While studying the latter work he made notes from which there eventually emerged his first mathematical memoir, entitled, "On certain theorems in the calculus of variations." By the same works his attention was attracted to the transformation of homogeneous functions by linear substitutions, and in the course of his subsequent investigations he was led to results which are now regarded as the foundation of the modern Higher Algebra. In the publication of his results he received friendly assistance from D. F. Gregory, a younger member of the Cambridge school, and editor of the newly founded *Cambridge Mathematical Journal*. Gregory and other friends suggested that Boole should take the regular mathematical course at Cambridge, but this he was unable to do; he continued to teach school for his own support and that of his aged parents, and to cultivate mathematical analysis in the leisure left by a laborious occupation.

Duncan F. Gregory was one of a Scottish family already distinguished in the annals of science. His grandfather was James Gregory, the inventor of the refracting telescope and discoverer of a convergent series for π. A cousin of his father was David Gregory, a special friend and fellow worker of Sir Isaac Newton. D. F. Gregory graduated at Cambridge, and after graduation he immediately turned his attention to the logical foundations of analysis. He had before him Peacock's theory of algebra, and he knew that in the analysis as developed by the French school there were many remarkable phenomena awaiting explanation; particularly theorems which involved what was called the separation of symbols. He embodied his results in a paper "On the real Nature of symbolical Algebra" which was printed in the *Transactions* of the Royal Society of Edinburgh.

Boole became a master of the method of separation of symbols, and by attempting to apply it to the solution of differential equations with variable coefficients was led to devise a general method in analysis. The account of it was printed in the *Transactions* of the Royal Society of London, and brought its author a Royal medal. Boole's study of the separation of symbols naturally led him to a study of the foundations of analysis, and he had before him the writings of Peacock, Gregory and De Morgan. He was led to entertain very wide views of the domain of mathematical analysis; in fact that it was coextensive with exact analysis, and so embraced formal logic. In 1848, as we have seen, the controversy arose between Hamilton and De Morgan about the quantification of terms; the general interest which that controversy awoke in the relation of mathematics to logic induced Boole to prepare for publication his views on the subject, which he did that same year in a small volume entitled *Mathematical Analysis of Logic*.

About this time what are denominated the Queen's Colleges of Ireland were instituted at Belfast, Cork and Galway; and in 1849 Boole was appointed to the chair of mathematics in the Queen's College at Cork. In this more suitable environment he set himself to the preparation of a more elaborate work on the mathematical analysis of logic. For this purpose he read extensively books on psychology and logic, and as a result published in 1854 the work on which his fame chiefly rests—"An Investigation of the Laws of Thought, on which are founded the mathematical theories of logic and probabilities." Subsequently he prepared textbooks on *Differential Equations* and *Finite Differences*; the former of which remained the best English textbook on its subject until the publication of Forsyth's *Differential Equations*.

Prefixed to the *Laws of Thought* is a dedication to Dr. Ryall, Vice-President and Professor of Greek in the same College. In the following year, perhaps as a result of the dedication, he married Miss Everest, the niece of that colleague. Honors came: Dublin University made him an LL.D., Oxford a D.C.L.; and the Royal Society of London elected him a Fellow. But Boole's career was cut short in the midst of his usefulness and scientific labors. One day in 1864 he walked from his residence to the College, a distance of two miles, in a drenching rain, and lectured in wet clothes. The result was a feverish cold which soon fell upon his lungs and terminated his career on December 8, 1864, in the 50th year of his age.

De Morgan was the man best qualified to judge of the value of Boole's work in the field of logic; and he gave it generous praise and help. In writing to the Dublin Hamilton he said, "I shall be glad to see his work (*Laws of Thought*) out, for he has, I think, got hold of the true connection of algebra and logic." At another time he wrote to the same as follows: "All metaphysicians except you and I and Boole consider mathematics as four books of Euclid and algebra up to quadratic equations." We might infer that these three contemporary mathematicians who were likewise philosophers would form a triangle of friends. But it was not so; Hamilton was a friend of De Morgan, and De Morgan a friend of Boole; but the relation of *friend*, although convertible, is not necessarily transitive. Hamilton met De Morgan only once in his life, Boole on the other hand with comparative frequency; yet he had a voluminous correspondence with the former extending over 20 years, but almost no correspondence with the latter. De Morgan's investigations of double algebra and triple algebra prepared him to appreciate the quaternions, whereas Boole was too much given over to the symbolic theory to appreciate geometric algebra.

Hamilton's biography has appeared in three volumes, prepared by his friend Rev. Charles Graves; De Morgan's biography has appeared in one volume, prepared by his widow; of Boole no biography has appeared. A biographical notice of Boole was written for the *Proceedings* of the Royal Society of London by his friend the Rev. Robert Harley, and it is to it that I am indebted for most of my biographical data. Last summer when in England I learned that the reason why no adequate biography of Boole had appeared was the unfortunate temper and lack of sound judgment of his widow. Since her husband's death Mrs. Boole has published a paradoxical book of the false kind worthy of a notice in De Morgan's *Budget*.

The work done by Boole in applying mathematical analysis to logic necessarily led him to consider the general question of how reasoning is accomplished by means of symbols. The view which he adopted on this point is stated at page 68 of the *Laws of Thought*. "The conditions of valid reasoning by the aid of symbols, are: *First*, that a fixed interpretation be assigned to the symbols employed in the expression of the data; and that the laws of the combination of those symbols be correctly determined from that interpretation; *Second*, that the formal processes of solution or demonstration be conducted throughout in obedience to all the laws determined as above, without regard to the question of the interpretability of the particular results obtained; *Third*, that the final result be interpretable in form, and that it be actually interpreted in accordance with that system of interpretation which has been employed in the expression of the data." As regards these conditions it may be observed that they are very different from the formalist view of Peacock and De Morgan, and that they incline towards a realistic view of analysis, as held by Hamilton. True he speaks of interpretation instead of meaning, but it is a fixed interpretation; and the rules for the processes of solution are not to be chosen arbitrarily, but are to be found out from the particular system of interpretation of the symbols.

It is Boole's second condition which chiefly calls for study and examination; respecting it he observes as follows: "The principle in question may be considered as resting upon a general law of the mind, the knowledge of which is not given to us *a priori*, that is, antecedently to experience, but is derived, like the knowledge of the other laws of the mind, from the clear manifestation of the general principle in the particular instance. A single example of reasoning, in which symbols are employed in obedience to laws founded upon their interpretation, but without any sustained reference to that interpretation, the chain of demonstration conducting us through intermediate steps which are not interpretable to a final result which is interpretable, seems not only to establish the validity of the particular application, but to make known to us the general law manifested therein. No accumulation of instances can properly add weight to such evidence. It may furnish us with clearer conceptions of that common element of truth upon which the application of the principle depends, and so prepare the way for its reception. It may, where the immediate force of the evidence is not felt, serve as a verification, *a posteriori*, of the practical validity of the principle in question. But this does not affect the position affirmed, viz., that the general principle must be seen in the particular instance—seen to be general in application as well as true in the special example. The employment of the uninterpretable symbol $\sqrt{-1}$ in the intermediate processes of trigonometry furnishes an illustration of what has been said. I apprehend that there is no mode of explaining that application which does not covertly assume the very principle in question. But that principle, though not, as I conceive, warranted by formal reasoning based upon other grounds, seems to deserve a place among those axiomatic truths which constitute in some sense the foundation of general knowledge, and which may properly be regarded as expressions of the mind's own laws and constitution."

We are all familiar with the fact that algebraic reasoning may be conducted through intermediate equations without requiring a sustained reference to the meaning of these equations; but it is paradoxical to say that these equations can, in any case, have no meaning or interpretation. It may not be necessary to consider their meaning, it may even be difficult to find their meaning, but that they have a meaning is a dictate of common sense. It is entirely paradoxical to say that, as a general process, we can start from equations having a meaning, and arrive at equations having a meaning by passing through equations which have no meaning. The particular instance in which Boole sees the truth of the paradoxical principle is the successful employment of the uninterpretable symbol $\sqrt{-1}$ in the intermediate processes of trigonometry. So soon then as this symbol is interpreted, or rather, so soon as its meaning is demonstrated, the evidence for the principle fails, and Boole's transcendental logic falls.

In the algebra of quantity we start from elementary symbols denoting numbers, but are soon led to compound forms which do not reduce to numbers; so in the algebra of logic we start from elementary symbols denoting classes, but are soon introduced to compound expressions which cannot be reduced to simple classes. Most mathematical logicians say, Stop, we do not know what this combination means. Boole says, It may be meaningless, go ahead all the same. The design of the *Laws of Thought* is stated by the author to be to investigate the fundamental laws of those operations of the mind by which reasoning is performed; to give expression to them in the symbolical language of a Calculus, and upon this foundation to establish the Science of Logic and construct its method; to make that method itself the basis of a general method for the application of the mathematical doctrine of Probabilities; and, finally to collect from the various elements of truth brought to view in the course of these inquiries some probable intimations concerning the nature and constitution of the human mind.

Boole's inventory of the symbols required in the algebra of logic is as follows: *first*, Literal symbols, as x, y, etc., representing things as subjects of our conceptions; *second*, Signs of operation, as $+$, $-$, $\times$, standing for those operations of the mind by which the conceptions of things are combined or resolved so as to form new conceptions involving the same elements; *third*, The sign of identity $=$; not equality merely, but identity which involves equality. The symbols x, y, etc., are used to denote classes; and it is one of Boole's maxims that substantives and adjectives alike denote classes. "They may be regarded," he says, "as differing only in this respect, that the former expresses the substantive existence of the individual thing or things to which it refers, the latter implies that existence. If we attach to the adjective the universally understood subject, 'being' or 'thing,' it becomes virtually a substantive, and may for all the essential purposes of reasoning be replaced by the substantive." Let us then agree to represent the class of individuals to which a particular name is applicable by a single letter as x. If the name is *men* for instance, let x represent *all men*, or the class *men.* Again, if an adjective, as *good*, is employed as a term of description, let us represent by a letter, as y, all things to which the description *good* is applicable, that is, *all good things* or the class *good things.* Then the combination yx will represent *good men.*

Boole's symbolic logic was brought to my notice by Professor Tait, when I was a student in the physical laboratory of Edinburgh University. I studied the *Laws of Thought* and I found that those who had written on it regarded the method as highly mysterious; the results wonderful, but the processes obscure. I reduced everything to diagram and model, and I ventured to publish my views on the subject in a small volume called *Principles of the Algebra of Logic*; one of the chief points I made is the philological and analytical difference between the substantive and the adjective. What I said was that the word *man* denotes a class, but the word *white* does not; in the former a definite unit-object is specified, in the latter no unit-object is specified. We can exhibit a type of a *man*, we cannot exhibit a type of a *white.*

The identification of the substantive and adjective on the one hand and their discrimination on the other hand, lead to different conceptions of what De Morgan called the *universe.* Boole's conception of the Universe is as follows (*Laws of Thought*, p. 42): "In every discourse, whether of the mind conversing with its own thoughts, or of the individual in his intercourse with others, there is an assumed or expressed limit within which the subjects of its operation are confined. The most unfettered discourse is that in which the words we use are understood in the widest possible application, and for them the limits of discourse are coextensive with those of the universe itself. But more usually we confine ourselves to a less spacious field. Sometimes in discoursing of men we imply (without expressing the limitation) that it is of men only under certain circumstances and conditions that we speak, as of civilized men, or of men in the vigor of life, or of men under some other condition or relation. Now, whatever may be the extent of the field within which all the objects of our discourse are found, that field may properly be termed the universe of discourse."

Another view leads to the conception of the Universe as a collection of homogeneous units, which may be finite or infinite in number; and in a particular problem the mind considers the relation of identity between different groups of this collection. This *universe* corresponds to *the series of events*, in the theory of Probability; and the characters correspond to the different ways in which the event may happen. The difference is that the Algebra of Logic considers necessary data and relations; while the theory of Probability considers probable data and relations. I will explain the elements of Boole's method on this theory.

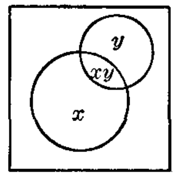

FIG. 1.

The square is a collection of points: it may serve to represent any collection of homogeneous units, whether finite or infinite in number, that is, the universe of the problem. Let x denote *inside the left-hand circle*, and y *inside the right-hand circle*. Uxy will denote the points inside both circles (Fig. 1). In arithmetical value x may range from 1 to 0; so also y; while xy cannot be greater than x or y, or less than 0 or $x + y - 1$. This last is the principle of the syllogism. From the co-ordinate nature of the operations x and y, it is evident that $Uxy = Uyx$; but this is a different thing from commuting, as Boole does, the relation of U and x, which is not that of co-ordination, but of subordination of x to U, and which is properly denoted by writing U first.

Suppose y to be the same character as x; we will then always have $Uxx = Ux$; that is, an elementary selective symbol x is always such that $x^2 = x$. These are but the symbols of ordinary algebra which satisfy this relation, namely 1 and 0; these are also the extreme selective symbols *all* and *none*. The law in question was considered Boole's paradox; it plays a very great part in the development of his method.

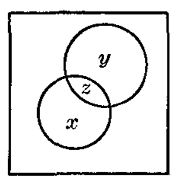

FIG. 2.

Let $Uxy = Uz$, where $=$ means *identical with*, not *equal to*; we may write $xy = z$, leaving the U to be understood. It does not mean that the combination of characters xy is identical with the character z; but that those points which have the characters x and y are identical with the points which have the character z (Fig. 2). From $xy = z$, we derive $x = \frac{1}{y}z$; what is the meaning of this expression? We shall return to the question, after we have considered $+$ and $-$.

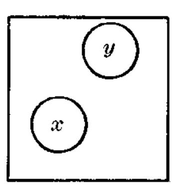

FIG. 3

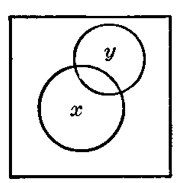

FIG. 4

Let us now consider the expression $U(x+y)$. If the x points and the y points are outside of one another, it means the sum of the x points and the y points (Fig. 3). So far all are agreed. But suppose that the x points and the y points are partially identical (Fig. 4); then there arises difference of opinion. Boole held that the common points must be taken twice over, or in other words that the symbols x and y must be treated all the same as if they were independent of one another; otherwise, he held, no general analysis is possible. $U(x+y)$ will not in general denote a single class of points; it will involve in general a duplication.

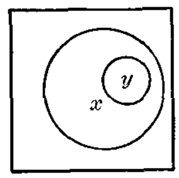

FIG. 5.

Similarly, Boole held that the expression $U(x-y)$ does not involve the condition of the Uy being wholly included in the Ux (Fig. 5). If that condition is satisfied, $U(x-y)$ denotes a simple class; namely, the Ux's *without* the Uy's. But when there is partial coincidence (as in Fig. 4), the common points will be cancelled, and the result will be the Ux's which are not y taken positively and the Uy's which are not x taken negatively. In Boole's view $U(x-y)$ was in general an intermediate uninterpretable form, which might be used in reasoning the same way as analysts used $\sqrt{-1}$.

Most of the mathematical logicians who have come after Boole are men who would have stuck at the impossible subtraction in ordinary algebra. They say virtually, "How can you throw into a heap the same things twice over; and how can you take from a heap things that are not there." Their great principle is the impossibility of taking the pants from a Highlander. Their only conception of the analytical processes of addition and subtraction is throwing into a heap and taking out of a heap. It does not occur to them that the processes of algebra are *ideal*, and not subject to gross material restrictions.

If $x+y$ denotes a quality without duplication, it will satisfy the condition

$$\begin{aligned} (x+y)^2 &= x+y, \\ x^2+2xy+y^2 &= x+y, \\ \text{but } x^2 = x, y^2 &= y, \\ \therefore 2xy &= 0. \end{aligned}$$

Similarly, if $x - y$ denote a simple quality, then

$$\begin{aligned} (x-y)^2 &= x - y, \\ x^2 + y^2 - 2xy &= x - y, \\ x^2 = x, \quad y^2 &= y, \\ \text{therefore, } y - 2xy &= -y, \\ \therefore y &= xy. \end{aligned}$$

In other words, the Uy must be included in the Ux (Fig. 5). Here we have assumed that the law of signs is the same as in ordinary algebra, and the result comes out correct.

Suppose Uz=Uxy; then $Ux = U\frac{1}{y}z$. How are the Ux's related to the Uy's, and the Uz's? From the diagram (in Fig. 2) we see that the Ux's are identical with all the Uyz's together with an indefinite portion of the U's, which are neither y nor z. Boole discovered a general method for finding the meaning of any function of elementary logical symbols, which applied to the above case, is as follows:

When y is an elementary symbol,

$$\begin{aligned} 1 &= y + (1-y). \\ \text{Similarly } 1 &= z + (1-z). \\ \therefore 1 &= yz + y(1-z) + (1-y)z + (1-y)(1-z), \end{aligned}$$

which means that the U's either have both qualities y and z, or y but not z, or z but not y, or neither y and z. Let

$$\frac{1}{y}z = Ayz + By(1-z) + C(1-y)z + D(1-y)(1-z),$$

it is required to determine the coefficients A, B, C, D. Suppose $y = 1$, $z = 1$; then $1 = A$. Suppose $y = 1$, $z = 0$, then $0 = B$. Suppose $y = 0$, $z = 1$; then $\frac{1}{0} = C$, and C is infinite; therefore $(1-y)z = 0$; which we see to be true from the diagram. Suppose $y = 0$, $z = 0$; then $\frac{0}{0} = D$, or D is indeterminate. Hence

$$\frac{1}{y}z = yz + \text{an indefinite portion of } (1-y)(1-z).$$

* * * * *

Boole attached great importance to the index law $x^2 = x$. He held that it expressed a law of thought, and formed the characteristic distinction of the operations of the mind in its ordinary discourse and reasoning, as compared with its operations when occupied with the general algebra of quantity. It makes possible, he said, the solution of a quintic or equation of higher degree, when the symbols are logical. He deduces from it the axiom of metaphysicians which is termed the principle of contradiction, and which affirms that it is impossible for any being to possess a quality, and at the same time not to possess it. Let x denote an elementary quality applicable to the universe U; then $1 - x$ denotes the absence of that quality. But if $x^2 = x$, then $0 = x - x^2, 0 = x(1 - x)$, that is, from $Ux^2 = Ux$ we deduce $Ux(1 - x) = 0$.

He considers $x(1 - x) = 0$ as an expression of the principle of contradiction. He proceeds to remark: "The above interpretation has been introduced not on account of its immediate value in the present system, but as an illustration of a significant fact in the philosophy of the intellectual powers, viz., that what has been commonly regarded as the fundamental axiom of metaphysics is but the consequence of a law of thought, mathematical in its form. I desire to direct attention also to the circumstance that the equation in which that fundamental law of thought is expressed is an equation of the second degree. Without speculating at all in this chapter upon the question whether that circumstance is necessary in its own nature, we may venture to assert that if it had not existed, the whole procedure of the understanding would have been different from what it is."

We have seen that De Morgan investigated long and published much on mathematical logic. His logical writings are characterized by a display of many symbols, new alike to logic and to mathematics; in the words of Sir W. Hamilton of Edinburgh, they are "horrent with mysterious spiculæ." It was the great merit of Boole's work that he used the immense power of the ordinary algebraic notation as an exact language and proved its power for making ordinary language more exact. De Morgan could well appreciate the magnitude of the feat, and he gave generous testimony to it as follows:

"Boole's system of logic is but one of many proofs of genius and patience combined. I might legitimately have entered it among my *paradoxes*, or things counter to general opinion: but it is a paradox which, like that of Copernicus, excited admiration from its first appearance. That the symbolic processes of algebra, invented as tools of numerical calculation, should be competent to express every act of thought, and to furnish the grammar and dictionary of an all-containing system of logic, would not have been believed until it was proved. When Hobbes, in the time of the Commonwealth, published his "Computation or Logique" he had a remote glimpse of some of the points which are placed in the light of day by Mr. Boole. The unity of the forms of thought in all the applications of reason, however remotely separated, will one day be matter of notoriety and common wonder: and Boole's name will be remembered in connection with one of the most important steps towards the attainment of this knowledge."