Chapter 5

ARTHUR CAYLEY[1]

(1821-1895)

Arthur Cayley was born at Richmond in Surrey, England, on August 16, 1821. His father, Henry Cayley, was descended from an ancient Yorkshire family, but had settled in St. Petersburg, Russia, as a merchant. His mother was Maria Antonia Doughty, a daughter of William Doughty; who, according to some writers, was a Russian; but her father's name indicates an English origin. Arthur spent the first eight years of his life in St. Petersburg. In 1829 his parents took up their permanent abode at Blackheath, near London; and Arthur was sent to a private school. He early showed great liking for, and aptitude in, numerical calculations. At the age of 14 he was sent to King's College School, London; the master of which, having observed indications of mathematical genius, advised the father to educate his son, not for his own business, as he had at first intended, but to enter the University of Cambridge.

At the unusually early age of 17 Cayley began residence at Trinity College, Cambridge. As an undergraduate he had generally the reputation of being a mere mathematician; his chief diversion was novel-reading. He was also fond of travelling and mountain climbing, and was a member of the Alpine Club. The cause of the Analytical Society had now triumphed, and the *Cambridge Mathematical Journal* had been instituted by Gregory and Leslie Ellis. To this journal, at the age of twenty, Cayley contributed three papers, on subjects which had been suggested by reading the *Mécanique analytique* of Lagrange and some of the works of Laplace. We have already noticed how the works of Lagrange and Laplace served to start investigation in Hamilton and Boole. Cayley finished his undergraduate course by winning the place of Senior Wrangler, and the first Smith's prize. His next step was to take the M.A. degree, and win a Fellowship by competitive examination. He continued to reside at Cambridge for four years; during which time he took some pupils, but his main work was the preparation of 28 memoirs to the *Mathematical Journal*. On account of the limited tenure of his fellowship it was necessary to choose a profession; like

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De Morgan, Cayley chose the law, and at 25 entered at Lincoln's Inn, London. He made a specialty of conveyancing and became very skilled at the work; but he regarded his legal occupation mainly as the means of providing a livelihood, and he reserved with jealous care a due portion of his time for mathematical research. It was while he was a pupil at the bar that he went over to Dublin for the express purpose of hearing Hamilton's lectures on Quaternions. He sat alongside of Salmon (now provost of Trinity College, Dublin) and the readers of Salmon's books on Analytical Geometry know how much their author was indebted to his correspondence with Cayley in the matter of bringing his textbooks up to date. His friend Sylvester, his senior by five years at Cambridge, was then an actuary, resident in London; they used to walk together round the courts of Lincoln's Inn, discussing the theory of invariants and covariants. During this period of his life, extending over fourteen years, Cayley produced between two and three hundred papers.

At Cambridge University the ancient professorship of pure mathematics is denominated the Lucasian, and is the chair which was occupied by Sir Isaac Newton. About 1860 certain funds bequeathed by Lady Sadleir to the University, having become useless for their original purpose, were employed to establish another professorship of pure mathematicas, called the Sadlerian. The duties of the new professor were defined to be "to explain and teach the principles of pure mathematics and to apply himself to the advancement of that science." To this chair Cayley was elected when 42 years old. He gave up a lucrative practice for a modest salary; but he never regretted the exchange, for the chair at Cambridge enabled him to end the divided allegiance between law and mathematics, and to devote his energies to the pursuit which he liked best. He at once married and settled down in Cambridge. More fortunate than Hamilton in his choice, his home life was one of great happiness. His friend and fellow investigator, Sylvester, once remarked that Cayley had been much more fortunate than himself; that they both lived as bachelors in London, but that Cayley had married and settled down to a quiet and peaceful life at Cambridge; whereas he had never married, and had been fighting the world all his days. The remark was only too true (as may be seen in the lecture on Sylvester).

At first the teaching duty of the Sadlerian professorship was limited to a course of lectures extending over one of the terms of the academic year; but when the University was reformed about 1886, and part of the college funds applied to the better endowment of the University professors, the lectures were extended over two terms. For many years the attendance was small, and came almost entirely from those who had finished their career of preparation for competitive examinations; after the reform the attendance numbered about fifteen. The subject lectured on was generally that of the memoir on which the professor was for the time engaged.

The other duty of the chair—the advancement of mathematical science was—discharged in a handsome manner by the long series of memoirs which he published, ranging over every department of pure mathematics. But it was also discharged in a much less obtrusive way; he became the standing referee on the merits of mathematical papers to many societies both at home and abroad.

Many mathematicians, of whom Sylvester was an example, find it irksome to study what others have written, unless, perchance, it is something dealing directly with their own line of work. Cayley was a man of more cosmopolitan spirit; he had a friendly sympathy with other workers, and especially with young men making their first adventure in the field of mathematical research. Of referee work he did an immense amount; and of his kindliness to young investigators I can speak from personal experience. Several papers which I read before the Royal Society of Edinburgh on the Analysis of Relationships were referred to him, and he recommended their publication. Soon after I was invited by the Anthropological Society of London to address them on the subject, and while there, I attended a meeting of the Mathematical Society of London. The room was small, and some twelve mathematicians were assembled round a table, among whom was Prof. Cayley, as became evident to me from the proceedings. At the close of the meeting Cayley gave me a cordial handshake and referred in the kindest terms to my papers which he had read. He was then about 60 years old, considerably bent, and not filling his clothes. What was most remarkable about him was the active glance of his gray eyes and his peculiar boyish smile.

In 1876 he published a *Treatise on Elliptic Functions*, which was his only book. He took great interest in the movement for the University education of women. At Cambridge the women's colleges are Girton and Newnham. In the early days of Girton College he gave direct help in teaching, and for some years he was chairman of the council of Newnham College, in the progress of which he took the keenest interest to the last. His mathematical investigations did not make him a recluse; on the contrary he was of great practical usefulness, especially from his knowledge of law, in the administration of the University.

In 1872 he was made an honorary fellow of Trinity College, and three years later an ordinary fellow, which meant stipend as well as honor. About this time his friends subscribed for a presentation portrait, which now hangs on the side wall of the dining hall of Trinity College, next to the portrait of James Clerk Maxwell, while on the end wall, behind the high table, hang the more ancient portraits of Sir Isaac Newton and Lord Bacon of Verulam. In the portrait Cayley is represented as seated at a desk, quill in hand, after the mode in which he used to write out his mathematical investigations. The investigation, however, was all thought out in his mind before he took up the quill.

Maxwell was one of the greatest electricians of the nineteenth century. He was a man of philosophical insight and poetical power, not unlike Hamilton, but differing in this, that he was no orator. In that respect he was more like Goldsmith, who "could write like an angel, but only talked like poor poll." Maxwell wrote an address to the committee of subscribers who had charge of the Cayley portrait fund, wherein the scientific poet with his pen does greater honor to the mathematician than the artist, named Dickenson, could do with his brush. Cayley had written on space of n dimensions, and the main point in the address is derived from the artist's business of depicting on a plane what exists in space:

O wretched race of men, to space confined!

What honor can ye pay to him whose mind
To that which lies beyond hath penetrated?
The symbols he hath formed shall sound his praise,
And lead him on through unimagined ways
To conquests new, in worlds not yet created.

First, ye Determinants, in ordered row
And massive column ranged, before him go,
To form a phalanx for his safe protection.
Ye powers of the nth root of -1!
Around his head in endless cycles run,
As unembodied spirits of direction.

And you, ye undevelopable scrolls!
Above the host where your emblazoned rolls,
Ruled for the record of his bright inventions.
Ye cubic surfaces! by threes and nines
Draw round his camp your seven and twenty lines
The seal of Solomon in three dimensions.

March on, symbolic host! with step sublime,
Up to the flaming bounds of Space and Time!
There pause, until by Dickenson depicted
In two dimensions, we the form may trace
Of him whose soul, too large for vulgar space,
In n dimensions flourished unrestricted.

The verses refer to the subjects investigated in several of Cayley's most elaborate memoirs; such as, Chapters on the Analytical Geometry of n dimensions; On the theory of Determinants; Memoir on the theory of Matrices; Memoirs on skew surfaces, otherwise Scrolls; On the delineation of a Cubic Scroll, etc.

In 1881 he received from the Johns Hopkins University, Baltimore, where Sylvester was then professor of mathematics, an invitation to deliver a course of lectures. He accepted the invitation, and lectured at Baltimore during the first five months of 1882 on the subject of the *Abelian and Theta Functions*.

The next year Cayley came prominently before the world, as President of the British Association for the Advancement of Science. The meeting was held at Southport, in the north of England. As the President's address is one of the great popular events of the meeting, and brings out an audience of general culture, it is usually made as little technical as possible. Hamilton was the kind of mathematician to suit such an occasion, but he never got the office, on account of his occasional breaks. Cayley had not the oratorical, the philosophical, or the poetical gifts of Hamilton, but then he was an eminently safe man. He took for his subject the Progress of Pure Mathematics; and he opened his address in the following naïve manner: "I wish to speak to you to-night upon Mathematics. I am quite aware of the difficulty arising from the abstract nature of my subject; and if, as I fear, many or some of you, recalling the providential addresses at former meetings, should wish that you were now about to have from a different President a discourse on a different subject, I can very well sympathize with you in the feeling. But be that as it may, I think it is more respectful to you that I should speak to you upon and do my best to interest you in the subject which has occupied me, and in which I am myself most interested. And in another point of view, I think it is right that the address of a president should be on his own subject, and that different subjects should be thus brought in turn before the meetings. So much the worse, it may be, for a particular meeting: but the meeting is the individual, which on evolution principles, must be sacrificed for the development of the race." I daresay that after this introduction, all the evolution philosophers listened to him attentively, whether they understood him or not. But Cayley doubtless felt that he was addressing not only the popular audience then and there before him, but the mathematicians of distant places and future times; for the address is a valuable historical review of various mathematical theories, and is characterized by freshness, independence of view, suggestiveness, and learning.

In 1889 the Cambridge University Press requested him to prepare his mathematical papers for publication in a collected form—a request which he appreciated very much. They are printed in magnificent quarto volumes, of which seven appeared under his own editorship. While editing these volumes, he was suffering from a painful internal malady, to which he succumbed on January 26, 1895, in the 74th year of his age. When the funeral took place, a great assemblage met in Trinity Chapel, comprising members of the University, official representatives of Russia and America, and many of the most illustrious philosophers of Great Britain.

The remainder of his papers were edited by Prof. Forsyth, his successor in the Sadlerian chair. The Collected Mathematical papers number thirteen quarto volumes, and contain 967 papers. His writings are his best monument, and certainly no mathematician has ever had his monument in grander style. De Morgan's works would be more extensive, and much more useful, but he did not have behind him a University Press. As regards fads, Cayley retained to the last his fondness for novel-reading and for travelling. He also took special pleasure in paintings and architecture, and he practised water-color painting, which he found useful sometimes in making mathematical diagrams.

To the third edition of Tait's *Elementary Treatise on Quaternions*, Cayley contributed a chapter entitled "Sketch of the analytical theory of quaternions." In it the $\sqrt{-1}$ reappears in all its glory, and in entire, so it is said, independence of i, j, k. The remarkable thing is that Hamilton started with a quaternion theory of analysis, and that Cayley should present instead an analytical theory of quaternions. I daresay that Prof. Tait was sorry that he allowed the chapter to enter his book, for in 1894 there arose a brisk discussion between himself and Cayley on "Coordinates versus Quaternions," the record of which is printed in the Proceedings of the Royal Society of Edinburgh. Cayley maintained the position that while coordinates are applicable to the whole science of geometry and are the natural and appropriate basis and method in the science, quaternions seemed a particular and very artificial method for treating such parts of the science of three-dimensional geometry as are most naturally discussed by means of the rectangular coordinates x, y, z. In the course of his paper Cayley says: "I have the highest admiration for the notion of a quaternion; but, as I consider the full moon far more beautiful than any moonlit view, so I regard the notion of a quaternion as far more beautiful than any of its applications. As another illustration, I compare a quaternion formula to a pocket-map—a capital thing to put in one's pocket, but which for use must be unfolded: the formula, to be understood, must be translated into coordinates." He goes on to say, "I remark that the imaginary of ordinary algebra—for distinction call this θ—has no relation whatever to the quaternion symbols i, j, k; in fact, in the general point of view, all the quantities which present themselves, are, or may be, complex values $a + \theta b$, or in other words, say that a scalar quantity is in general of the form $a+\theta b$. Thus quaternions do not properly present themselves in plane or two-dimensional geometry at all; but they belong essentially to solid or three-dimensional geometry, and they are most naturally applicable to the class of problems which in coordinates are dealt with by means of the three rectangular coordinates x, y, z."

To the pocketbook illustration it may be replied that a set of coordinates is an immense wall map, which you cannot carry about, even though you should roll it up, and therefore is useless for many important purposes. In reply to the arguments, it may be said, *first*, $\sqrt{-1}$ has a relation to the symbols i, j, k, for each of these can be analyzed into a unit axis multiplied by $\sqrt{-1}$; *second*, as regards plane geometry, the ordinary form of complex quantity is a degraded form of the quaternion in which the constant axis of the plane is left unspecified. Cayley took his illustrations from his experience as a traveller. Tait brought forward an illustration from which you might imagine he had visited the Bethlehem Iron Works, and hunted tigers in India. He says, "A much more natural and adequate comparison would, it seems to me, liken Coordinate Geometry to a steam-hammer, which an expert may employ on any destructive or constructive work of one general kind, say the cracking of an eggshell, or the welding of an anchor. But you must have your expert to manage it, for without him it is useless. He has to toil amid the heat, smoke, grime, grease, and perpetual din of the suffocating engine-room. The work has to be brought to the hammer, for it cannot usually be taken to its work. And it is not in general, transferable; for each expert, as a rule, knows, fully and confidently, the working details of his own weapon only. Quaternions, on the other hand, are like the elephant's trunk, ready at *any* moment for *anything*, be it to pick up a crumb or a field-gun, to strangle a tiger, or uproot a tree; portable in the extreme, applicable anywhere—alike in the trackless jungle and in the barrack square—directed by a little native who requires no special skill or training, and who can be transferred from one elephant to another without much hesitation. Surely this, which adapts itself to its work, is the grander instrument. But then, *it* is the natural, the other, the artificial one."

The reply which Tait makes, so far as it is an argument, is: There are two systems of quaternions, the i, j, k one, and another one which Hamilton developed from it; Cayley knows the first only, he himself knows the second; the former is an intensely artificial system of imaginaries, the latter is the natural organ of expression for quantities in space. Should a fourth edition of his *Elementary Treatise* be called for i, j, k will disappear from it, excepting in Cayley's chapter, should it be retained. Tait thus describes the first system: "Hamilton's extraordinary *Preface* to his first great book shows how from Double Algebras, through Triplets, Triads, and Sets, he finally reached Quaternions. This was the genesis of the Quaternions of the forties, and the creature thus produced is still essentially the Quaternion of Prof. Cayley. It is a magnificent analytical conception; but it is nothing more than the full development of the system of imaginaries i, j, k; defined by the equations,

$$i^2 = j^2 = k^2 = ijk = -1,$$

with the associative, but *not* the commutative, law for the factors. The novel and splendid points in it were the treatment of all directions in space as essentially alike in character, and the recognition of the unit vector's claim to rank also as a quadrantal versor. These were indeed inventions of the first magnitude, and of vast importance. And here I thoroughly agree with Prof. Cayley in his admiration. Considered as an analytical system, based throughout on pure imaginaries, the Quaternion method is elegant in the extreme. But, unless it had been also something more, something very different and much higher in the scale of development, I should have been content to admire it;—and to pass it by."

From "the most intensely artificial of systems, arose, as if by magic, an absolutely natural one" which Tait thus further describes. "To me Quaternions are primarily a Mode of Representation:—immensely superior to, but of essentially the same kind of usefulness as, a diagram or a model. They are, virtually, the thing represented; and are thus antecedent to, and independent of, coordinates; giving, in general, all the main relations, in the problem to which they are applied, without the necessity of appealing to coordinates at all. Coordinates may, however, easily be read into them:—when anything (such as metrical or numerical detail) is to be gained thereby. Quaternions, in a word, exist in space, and we have only to recognize them:—but we have to invent or imagine coordinates of all kinds."

To meet the objection why Hamilton did not throw i, j, k overboard, and expound the developed system, Tait says: "Most unfortunately, alike for himself and for his grand conception, Hamilton's nerve failed him in the composition of his first great volume. Had he then renounced, for ever, all dealings with i, j, k, his triumph would have been complete. He spared Agog, and the best of the sheep, and did not utterly destroy them. He had a paternal fondness for i, j, k; perhaps also a not unnatural liking for a meretricious title such as the mysterious word *Quaternion*; and, above all, he had an earnest desire to make the utmost return in his power for the liberality shown him by the authorities of Trinity College, Dublin. He had fully recognized, and proved to others, that his i, j, k, were mere excrescences and blots on his improved method:—but he unfortunately considered that their continued (if only partial) recognition was indispensable to the reception of his method by a world steeped in—Cartesianism! Through the whole compass of each of his tremendous volumes one can find traces of his desire to avoid even an allusion to i, j, k, and along with them, his sorrowful conviction that, should he do so, he would be left without a single reader."

To Cayley's presidential address we are indebted for information about the view which he took of the foundations of exact science, and the philosophy which commended itself to his mind. He quoted Plato and Kant with approval, J. S. Mill with faint praise. Although he threw a sop to the empirical philosophers at the beginning of his address, he gave them something to think of before he finished.

He first of all remarks that the connection of arithmetic and algebra with the notion of time is far less obvious than that of geometry with the notion of space; in which he, of course, made a hit at Hamilton's theory of Algebra as the science of pure time. Further on he discusses the theory directly, and concludes as follows: "Hamilton uses the term algebra in a very wide sense, but whatever else he includes under it, he includes all that in contradistinction to the Differential Calculus would be called algebra. Using the word in this restricted sense, I cannot myself recognize the connection of algebra with the notion of time; granting that the notion of continuous progression presents itself and is of importance, I do not see that it is in anywise the fundamental notion of the science. And still less can I appreciate the manner in which the author connects with the notion of time his algebraical couple, or imaginary magnitude, $a + b\sqrt{-1}$." So you will observe that doctors differ—Tait and Cayley—about the soundness of Hamilton's theory of couples. But it can be shown that a couple may not only be represented on a straight line, but actually means a portion of a straight line; and as a line is unidimensional, this favors the truth of Hamilton's theory.

As to the nature of mathematical science Cayley quoted with approval from an address of Hamilton's:

"These purely mathematical sciences of algebra and geometry are sciences of the pure reason, deriving no weight and no assistance from experiment, and isolated or at least isolable from all outward and accidental phenomena. The idea of order with its subordinate ideas of number and figure, we must not call innate ideas, if that phrase be defined to imply that all men must possess them with equal clearness and fulness; they are, however, ideas which seem to be so far born with us that the possession of them in any conceivable degree is only the development of our original powers, the unfolding of our proper humanity."

It is the aim of the evolution philosopher to reduce all knowledge to the empirical status; the only intuition he grants is a kind of instinct formed by the experience of ancestors and transmitted cumulatively by heredity. Cayley first takes him up on the subject of arithmetic: "Whatever difficulty be raisable as to geometry, it seems to me that no similar difficulty applies to arithmetic; mathematician, or not, we have each of us, in its most abstract form, the idea of number; we can each of us appreciate the truth of a proposition in numbers; and we cannot but see that a truth in regard to numbers is something different in kind from an experimental truth generalized from experience. Compare, for instance, the proposition, that the sun, having already risen so many times, will rise to-morrow, and the next day, and the day after that, and so on; and the proposition that even and odd numbers succeed each other alternately *ad infinitum*; the latter at least seems to have the characters of universality and necessity. Or again, suppose a proposition observed to hold good for a long series of numbers, one thousand numbers, two thousand numbers, as the case may be: this is not only no proof, but it is absolutely no evidence, that the proposition is a true proposition, holding good for all numbers whatever; there are in the Theory of Numbers very remarkable instances of propositions observed to hold good for very long series of numbers which are nevertheless untrue."

Then he takes him up on the subject of geometry, where the empiricist rather boasts of his success. "It is well known that Euclid's twelfth axiom, even in Playfair's form of it, has been considered as needing demonstration; and that Lobatschewsky constructed a perfectly consistent theory, wherein this axiom was assumed not to hold good, or say a system of non-Euclidean plane geometry. My own view is that Euclid's twelfth axiom in Playfair's form of it does not need demonstration, but is part of our notion of space, of the physical space of our experience—the space, that is, which we become acquainted with by experience, but which is the representation lying at the foundation of all external experience. Riemann's view before referred to may I think be said to be that, having *in intellectu* a more general notion of space (in fact a notion of non-Euclidean space), we learn by experience that space (the physical space of our experience) is, if not exactly, at least to the highest degree of approximation, Euclidean space. But suppose the physical space of our experience to be thus only approximately Euclidean space, what is the consequence which follows? *Not* that the propositions of geometry are only approximately true, but that they remain absolutely true in regard to that Euclidean space which has been so long regarded as being the physical space of our experience."

In his address he remarks that the fundamental notion which underlies and pervades the whole of modern analysis and geometry is that of imaginary magnitude in analysis and of imaginary space (or space as a *locus in quo* of imaginary points and figures) in geometry. In the case of two given curves there are two equations satisfied by the coordinates (x, y) of the several points of intersection, and these give rise to an equation of a certain order for the coordinate x or y of a point of intersection. In the case of a straight line and a circle this is a quadratic equation; it has two roots real or imaginary. There are thus two values, say of x, and to each of these corresponds a single value of y. There are therefore two points of intersection, viz., a straight line and a circle intersect always in two points, real or imaginary. It is in this way we are led analytically to the notion of imaginary points in geometry. He asks, What is an imaginary point? Is there in a plane a point the coordinates of which have given imaginary values? He seems to say No, and to fall back on the notion of an imaginary space as the *locus in quo* of the imaginary point.